

ON THE CAUCHY PROBLEM FOR THE INTEGRABLE CAMASSA-HOLM TYPE EQUATION WITH CUBIC NONLINEARITY

YING FU, GUILONG GUI, YUE LIU, AND CHANGZHENG QU

ABSTRACT. In this paper, we are concerned with the Cauchy problem for the integrable Camassa-Holm type equation with cubic nonlinearity. We establish the local well-posedness in a range of the Besov spaces and derive the blow-up scenario. With analytic initial data, we then show that its solutions are analytic in both variables, globally in space and locally in time. Finally, we give geometric descriptions to this integrable equation.

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1. INTRODUCTION

In this paper, we are concerned with the following Cauchy problem of the Camassa-Holm type equation with cubic nonlinearity,

$$\begin{cases} m_t + (u^2 - u_x^2)m_x + 2u_x m^2 = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), \quad m = u - u_{xx}, & x \in \mathbb{R}. \end{cases} \quad (1.1)$$

The equation in (1.1) was proposed by Olver and Rosenau [25] as a new generalization of integrable system by implementing a simple explicit algorithm based on the bi-Hamiltonian representation of the classically integrable system. In most cases, these new nonlinear systems are endowed with nonlinear dispersion, and thus support non-smooth soliton-like structures. Later, it was obtained again by Qiao [28] from the two-dimensional Euler equation. It was shown in [28] that the equation in (1.1) admits the Lax-pair and the Cauchy problem (1.1) may be solved by the inverse scattering transform method.

The equation in (1.1) is completely integrable and can be rewritten as the bi-Hamiltonian form [25], that is

$$m_t = -((u^2 - u_x^2)m)_x = J \frac{\delta H_0}{\delta m} = K \frac{\delta H_1}{\delta m},$$

where

$$J = -\partial m \partial^{-1} m \partial, \quad \text{and} \quad K = \partial^3 - \partial,$$

corresponding to the Hamiltonian

$$H_0 = 2 \int_{\mathbb{R}} m u \, dx,$$

and the Hamiltonian

$$H_1 = \frac{1}{4} \int_{\mathbb{R}} \left(u^4 + 2u^2 u_x^2 - \frac{1}{3} u_x^4 \right) dx.$$

It also admits the Lax pair [28], that is

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = U(m, \lambda) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = V(m, u, \lambda) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

where

$$U(m, \lambda) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \lambda m \\ -\frac{1}{2} \lambda m & \frac{1}{2} \end{pmatrix}, \quad m = u - u_{xx},$$

and

$$V(m, u, \lambda) = \begin{pmatrix} \lambda^{-2} + \frac{1}{2}(u^2 - u_x^2) & -\lambda^{-1}(u - u_x) - \frac{1}{2} \lambda m(u^2 - u_x^2) \\ \lambda^{-1}(u + u_x) + \frac{1}{2} \lambda m(u^2 - u_x^2) & -\lambda^{-2} - \frac{1}{2}(u^2 - u_x^2) \end{pmatrix}.$$

In addition, there exist a cusped soliton [28] to (1.1) defined by

$$u(t, x) = u(X) = \pm \left(2 - 3 \cosh^2 X + \left(\cosh X + \frac{1}{3} \right) \sqrt{3(3 \cosh X + 1)(\cosh X - 1)} \right),$$

where $X = \frac{x}{2} - \frac{11}{6}t$, and a so called "W/M" -shape-peakon soliton in the form [28]

$$u(t, x) = u(X) = 2 - 3 \cosh^2 X + \left(\cosh X + \frac{1}{3} \right) \sqrt{3(3 \cosh X + 1)(\cosh X - 1)},$$

where $X = \frac{|x - \frac{11}{3}t|}{2} - \ln 2$.

The Camassa-Holm (CH) equation [2, 14] defined by

$$m_t + um_x + 2u_x m = 0, \quad m = u - u_{xx}$$

has attracted much attention in the last twenty years because of its interesting properties: complete integrability, existence of peaked solitons and multi-peakons, geometric formulations [5, 11, 21, 22] and the presence of breaking waves (i.e. a solution that remains bounded while its slope becomes unbounded in finite time) [7, 8, 9, 10]. Note that the nonlinearity in the CH equation is quadratic. In contrast to the integrable modified KdV equation with a cubic nonlinearity, it is our interest to find an integrable CH-type equations with a cubic nonlinearity. Indeed, two integrable CH-type equations with cubic nonlinearity have been discovered recently. One is the so-called Novikov equation [24] and the second one is the equation in (1.1) [25]. The integrability, peaked solitons, well-posedness, blow up phenomena of the Novikov equation have been studied extensively, see the ref. [18, 19, 24, 30, 31].

The goal of the present paper is to establish qualitative results for the initial value problem (1.1). We first study the local well-posedness for the strong solutions to the Cauchy problem (1.1). The proof of the local well-posedness is inspired by the argument of approximate solutions by Danchin [12] in the study of the local well-posedness to the CH equation. However, one problematic issue is that we here deal with a higher order nonlinearity in the Besov spaces, making the proof of several required nonlinear estimates somewhat delicate. These difficulties are nevertheless

overcome by carefully estimates for each iterative approximation of solutions to (1.1). With the local well-posedness obtained in hand, we then present a precise blow-up scenario and a conservative property. We also prove the analyticity of its solutions $u = u(t, x)$ in both variables, with x in \mathbb{R} and t in an interval around zero, provided that the initial profile u_0 is an analytic function on the real line. Hence, this analytic result can be viewed as a Cauchy-Kowalevski theorem for (1.1). Finally, we give the geometric descriptions to this equation.

It is well known that the solutions of the KdV equation are analytic in the space variable for all time [32] but are not analytic in the time variable [20]. In contrast with the KdV equation, the solutions to the Hunter-Saxton (HS) and Camassa-Holm equations are analytic in both space and time variables for a short time [17, 30]. Like the CH and HS equations, we will show that solutions of the Cauchy problem (1.1) are analytic in both space and time variables.

The plan of the paper is as follows. In Section 2, some preliminary properties, which will be used later, are presented. The local well-posedness in the Besov spaces is established in Section 3. In Section 4, a blow-up scenario and a global conservative property of (1.1) will be derived. Section 5 is devoted to the study of the analyticity of the Cauchy problem (1.1) based on a contraction type argument in a suitably chosen scale of the Banach spaces. Such an approach to analytic regularity of solutions to Cauchy problem (1.1) was initiated by Ovsjannikov [26, 27] as an abstract Cauchy-Kowalevski theorem and later further developed by Nirenberg [23], Baouendi and Goulaouic [1] among others and subsequently applied to the Euler and Navier-Stokes equation. In Appendix A, the precise geometric descriptions of the equation in (1.1) are given.

Notation. In the following, for a given Banach space Z , we denote its norm by $\|\cdot\|_Z$. Since all space of functions are over \mathbb{R} , for simplicity, we drop \mathbb{R} in our notations of function spaces if there is no ambiguity. We denote $\mathcal{F}u$ or \hat{u} the Fourier transform of the function u .

2. PRELIMINARIES

In this section, we first present the Littlewood-Paley theory and the properties of the Besov-Sobolev spaces which will play a key role to prove local well-posedness for the Cauchy problem (1.1). Then we introduce a new space, whose properties are studied. In order to verify the analyticity of solutions to (1.1), the abstract Cauchy-Kowalevski theorem for identifying analyticity of the Cauchy problem is presented.

Proposition 2.1. [4] (*Littlewood-Paley decomposition*) Let $\mathcal{B} \triangleq \{\xi \in \mathbb{R}, |\xi| \leq \frac{4}{3}\}$ and $\mathcal{C} \triangleq \{\xi \in \mathbb{R}, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. There exist two radial functions $\chi \in C_c^\infty(\mathcal{B})$ and $\varphi \in C_c^\infty(\mathcal{C})$ such that

$$\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1, \quad \forall \quad \xi \in \mathbb{R}^d,$$

$$\begin{aligned} |q - q'| \geq 2 &\Rightarrow \text{Supp} \varphi(2^{-q}\cdot) \cap \text{Supp} \varphi(2^{-q'}\cdot) = \emptyset, \\ q \geq 1 &\Rightarrow \text{Supp} \chi(\cdot) \cap \text{Supp} \varphi(2^{-q}\cdot) = \emptyset, \end{aligned}$$

and

$$\frac{1}{3} \leq \chi(\xi)^2 + \sum_{q \geq 0} \varphi((2^{-q}\xi))^2 \leq 1, \quad \forall \quad \xi \in \mathbb{R}^d.$$

Furthermore, let $h \triangleq \mathcal{F}^{-1}\varphi$ and $\tilde{h} \triangleq \mathcal{F}^{-1}\chi$. Then the dyadic operators Δ_q and S_q can be defined as follows

$$\begin{aligned}\Delta_q f &\triangleq \varphi(2^{-q}D)f = 2^{qd} \int_{\mathbb{R}^d} h(2^q y) f(x-y) dy \quad \text{for } q \geq 0, \\ S_q f &\triangleq \chi(2^{-q}D)f = \sum_{-1 \leq k \leq q-1} \Delta_k f = 2^{qd} \int_{\mathbb{R}^d} \tilde{h}(2^q y) f(x-y) dy, \\ \Delta_{-1} f &\triangleq S_0 f \text{ and } \Delta_q f \triangleq 0 \quad \text{for } q \leq -2.\end{aligned}$$

Lemma 2.1. [4] (*Bernstein's inequality*) Let \mathcal{B} be a ball with center 0 in \mathbb{R}^d and \mathcal{C} a ring with center 0 in \mathbb{R}^d . A constant C exists so that, for any positive real number λ , any non negative integer k , any smooth homogeneous function σ of degree m and any couple of real numbers (a, b) with $b \geq a \geq 1$, there hold

$$\begin{aligned}\text{Supp } \hat{u} \subset \lambda \mathcal{B} &\Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^{k+1} \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a}; \\ \text{Supp } \hat{u} \subset \lambda \mathcal{C} &\Rightarrow C^{-1-k} \lambda^k \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^{1+k} \lambda^k \|u\|_{L^a}; \\ \text{Supp } \hat{u} \subset \lambda \mathcal{C} &\Rightarrow \|\sigma(D)u\|_{L^b} \leq C_{\sigma,m} \lambda^{m+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a}.\end{aligned}$$

for any function $u \in L^a$.

Definition 2.1. [4] (*Besov space*) Let $s \in \mathbb{R}, 1 \leq p, r \leq \infty$. The inhomogenous Besov space $B_{p,r}^s(\mathbb{R}^d)$ ($B_{p,r}^s$ for short) is defined by

$$B_{p,r}^s \triangleq \{f \in \mathcal{S}'(\mathbb{R}^d); \quad \|f\|_{B_{p,r}^s} < \infty\},$$

where

$$\|f\|_{B_{p,r}^s} \triangleq \begin{cases} \left(\sum_{q \in \mathbb{Z}} 2^{qsr} \|\Delta_q f\|_{L^p}^r \right)^{\frac{1}{r}}, & \text{for } r < \infty, \\ \sup_{q \in \mathbb{Z}} 2^{qs} \|\Delta_q f\|_{L^p}, & \text{for } r = \infty. \end{cases}$$

If $s = \infty$, $B_{p,r}^\infty \triangleq \bigcap_{s \in \mathbb{R}} B_{p,r}^s$.

Proposition 2.2. [12, 13] *The following properties hold.*

- i) *Density:* if $p, r < \infty$, then $\mathcal{S}(\mathbb{R}^d)$ is dense in $B_{p,r}^s(\mathbb{R}^d)$.
- ii) *Sobolev embeddings:* if $p_1 \leq p_2$ and $r_1 \leq r_2$, then $B_{p_1,r_1}^s \hookrightarrow B_{p_2,r_2}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}$. If $s_1 < s_2$, $1 \leq p \leq +\infty$ and $1 \leq r_1, r_2 \leq +\infty$, then the embedding $B_{p,r_2}^{s_2} \hookrightarrow B_{p,r_1}^{s_1}$ is locally compact.
- iii) *Algebraic properties:* for $s > 0$, $B_{p,r}^s \cap L^\infty$ is an algebra. Moreover, $(B_{p,r}^s \text{ is an algebra}) \iff (B_{p,r}^s \hookrightarrow L^\infty) \iff (s > \frac{d}{p} \text{ or } (s \geq \frac{d}{p} \text{ and } r = 1))$.
- iv) *Fatou property:* if $(u^{(n)})_{n \in \mathbb{N}}$ is a bounded sequence of $B_{p,r}^s$ which tends to u in \mathcal{S}' , then $u \in B_{p,r}^s$ and

$$\|u\|_{B_{p,r}^s} \leq \liminf_{n \rightarrow \infty} \|u^{(n)}\|_{B_{p,r}^s}.$$

- v) *Complex interpolation:* if $u \in B_{p,r}^s \cap B_{p,r}^{\tilde{s}}$ and $\theta \in [0, 1]$, $1 \leq p, r \leq \infty$, then $u \in B_{p,r}^{\theta s + (1-\theta)\tilde{s}}$ and $\|u\|_{B_{p,r}^{\theta s + (1-\theta)\tilde{s}}} \leq \|u\|_{B_{p,r}^s}^\theta \|u\|_{B_{p,r}^{\tilde{s}}}^{1-\theta}$.

- vi) Let $m \in \mathbb{R}$ and f be a S^m -multiplier (that is, $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is smooth and satisfies that for all multi-index α , there exists a constant C_α such that for any

$\xi \in \mathbb{R}^d$, $|\partial^\alpha f(\xi)| \leq C_\alpha(1 + |\xi|)^{m-|\alpha|}$.) Then for all $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, the operator $f(D)$ is continuous from $B_{p,r}^s$ to $B_{p,r}^{s-m}$.

Lemma 2.2. [12, 13] Suppose that $(p, r) \in [1, +\infty]^2$ and $s > -\frac{d}{p}$. Let v be a vector field such that ∇v belongs to $L^1([0, T]; B_{p,r}^{s-1})$ if $s > 1 + \frac{d}{p}$ or to $L^1([0, T]; B_{p,r}^{\frac{d}{p}} \cap L^\infty)$ otherwise. Suppose also that $f_0 \in B_{p,r}^s$, $F \in L^1([0, T]; B_{p,r}^s)$ and that $f \in L^\infty([0, T]; B_{p,r}^s) \cap C([0, T]; \mathcal{S}')$ solves the d -dimensional linear transport equations

$$(T) \quad \begin{cases} \partial_t f + v \cdot \nabla f = F, \\ f|_{t=0} = f_0. \end{cases}$$

Then there exists a constant C depending only on s, p and d such that the following statements hold:

1) If $r = 1$ or $s \neq 1 + \frac{d}{p}$, then

$$\|f\|_{B_{p,r}^s} \leq \|f_0\|_{B_{p,r}^s} + \int_0^t \|F(\tau)\|_{B_{p,r}^s} d\tau + C \int_0^t V'(\tau) \|f(\tau)\|_{B_{p,r}^s} d\tau,$$

or

$$\|f\|_{B_{p,r}^s} \leq e^{CV(t)} \left(\|f_0\|_{B_{p,r}^s} + \int_0^t e^{-CV(\tau)} \|F(\tau)\|_{B_{p,r}^s} d\tau \right) \quad (2.1)$$

hold, where $V(t) = \int_0^t \|\nabla v(\tau)\|_{B_{p,r}^{\frac{d}{p}} \cap L^\infty} d\tau$ if $s < 1 + \frac{d}{p}$ and $V(t) = \int_0^t \|\nabla v(\tau)\|_{B_{p,r}^{s-1}} d\tau$ else.

2) If $s \leq 1 + \frac{d}{p}$ and, in addition, $\nabla f_0 \in L^\infty$, $\nabla f \in L^\infty([0, T] \times \mathbb{R}^d)$ and $\nabla F \in L^1([0, T]; L^\infty)$, then

$$\begin{aligned} & \|f(t)\|_{B_{p,r}^s} + \|\nabla f(t)\|_{L^\infty} \\ & \leq e^{CV(t)} \left(\|f_0\|_{B_{p,r}^s} + \|\nabla f_0\|_{L^\infty} + \int_0^t e^{-CV(\tau)} (\|F(\tau)\|_{B_{p,r}^s} + \|\nabla F(\tau)\|_{L^\infty}) d\tau \right) \end{aligned}$$

with $V(t) = \int_0^t \|\nabla v(\tau)\|_{B_{p,r}^{\frac{d}{p}} \cap L^\infty} d\tau$.

3) If $f = v$, then for all $s > 0$, the estimate (2.1) holds with $V(t) = \int_0^t \|\partial_x u(\tau)\|_{L^\infty} d\tau$.

4) If $r < +\infty$, then $f \in C([0, T]; B_{p,r}^s)$. If $r = +\infty$, then $f \in C([0, T]; B_{p,1}^{s'})$ for all $s' < s$.

Lemma 2.3. [13] Let $(p, p_1, r) \in [1, +\infty]^3$. Assume that $s > -d \min\{\frac{1}{p_1}, \frac{1}{p'}\}$ with $p' \triangleq (1 - \frac{1}{p})^{-1}$. Let $f_0 \in B_{p,r}^s$ and $F \in L^1([0, T]; B_{p,r}^s)$. Let v be a time dependent vector field such that $v \in L^\rho([0, T]; B_{\infty,\infty}^{-M})$ for some $\rho > 1$, $M > 0$ and $\nabla v \in L^1([0, T]; B_{p_1,\infty}^{\frac{d}{p_1}} \cap L^\infty)$ if $s < 1 + \frac{d}{p_1}$, and $\nabla v \in L^1([0, T]; B_{p_1,r}^{s-1})$ if $s > 1 + \frac{d}{p_1}$ or $s = 1 + \frac{d}{p_1}$ and $r = 1$. Then the transport equations (T) has a unique solution $f \in L^\infty([0, T]; B_{p,r}^s) \cap (\cap_{s' < s} C([0, T]; B_{p,1}^{s'}))$ and the inequalities in Lemma 2.2 hold true. If, moreover, $r < \infty$, then we have $f \in C([0, T]; B_{p,r}^s)$.

Lemma 2.4. [4] (1-D Moser-type estimates) Assume that $1 \leq p, r \leq +\infty$, the following estimates hold:

(i) for $s > 0$, $\|fg\|_{B_{p,r}^s(\mathbb{R})} \leq C(\|f\|_{B_{p,r}^s(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})} + \|g\|_{B_{p,r}^s(\mathbb{R})} \|f\|_{L^\infty(\mathbb{R})});$

(ii) for $s_1 \leq \frac{1}{p}$, $s_2 > \frac{1}{p}$ ($s_2 \geq \frac{1}{p}$ if $r = 1$) and $s_1 + s_2 > 0$,

$$\|fg\|_{B_{p,r}^{s_1+s_2}(\mathbb{R})} \leq C\|f\|_{B_{p,r}^{s_1}(\mathbb{R})} \|g\|_{B_{p,r}^{s_2}(\mathbb{R})},$$

where C 's are constants independent of f and g .

In order to apply the contraction argument to the proof of the analytic regularity of the Cauchy problem (1.1), we need a suitable scale of Banach spaces which we proceed to describe below.

For any $s > 0$, we set

$$E_s = \left\{ u \in C^\infty(\mathbb{R}) : |||u|||_s = \sup_{k \in \mathbb{N}_0} \frac{s^k \|\partial^k u\|_{H^2}}{k!/(k+1)^2} < \infty \right\},$$

where $H^2(\mathbb{R})$ is the Sobolev space of order two on the real line and \mathbb{N}_0 is the set of nonnegative integers. One can easily verify that E_s equipped with the norm $|||\cdot|||_s$ is a Banach space and that, for any $0 < s' < s$, E_s is continuously embedded in $E_{s'}$ with

$$|||u|||_{s'} \leq |||u|||_s.$$

Another simple consequence of the definition is that any u in E_s is a real analytic function on \mathbb{R} . Crucial for our purposes is the fact that each E_s forms an algebra under pointwise multiplication of functions.

Lemma 2.5. [17] *Let $0 < s < 1$. There is a constant $c > 0$, independent of s , such that for any u and v in E_s we have*

$$|||uv|||_s \leq c |||u|||_s |||v|||_s.$$

Lemma 2.6. [17] *There is a constant $c > 0$ such that for any $0 < s' < s < 1$, we have*

$$|||\partial_x u|||_{s'} \leq \frac{c}{s-s'} |||u|||_s,$$

and

$$|||(1 - \partial_x^2)^{-1} u|||_{s'} \leq |||u|||_s, \quad |||(1 - \partial_x^2)^{-1} \partial_x u|||_{s'} \leq |||u|||_s.$$

The following theorem comes from [1].

Theorem 2.1. [1] *Let $\{X_s\}_{0 < s < 1}$ be a scale of decreasing Banach spaces, namely for any $s' < s$ we have $X_s \subset X_{s'}$ and $|||\cdot|||_{s'} \leq |||\cdot|||_s$. Consider the Cauchy problem*

$$\begin{cases} \frac{du}{dt} = F(t, u(t)), \\ u(0) = 0. \end{cases} \quad (2.2)$$

Let T, R and C be positive constants and assume that F satisfies the following conditions

1) *If for $0 < s' < s < 1$ the function $t \mapsto u(t)$ is holomorphic in $|t| < T$ and continuous on $|t| \leq T$ with values in X_s and*

$$\sup_{|t| \leq T} |||u(t)|||_s < R,$$

then $t \mapsto F(t, u(t))$ is a holomorphic function on $|t| < T$ with values in $X_{s'}$.

2) *For any $0 < s' < s < 1$ and any $u, v \in X_s$ with $|||u|||_s < R$, $|||v|||_s < R$,*

$$\sup_{|t| \leq T} |||F(t, u) - F(t, v)|||_{s'} \leq \frac{C}{s-s'} |||u - v|||_s.$$

3) *There exists $M > 0$ such that for any $0 < s < 1$,*

$$\sup_{|t| \leq T} |||F(t, 0)|||_s \leq \frac{M}{1-s}.$$

Then there exists a $T_0 \in (0, T)$ and a unique function $u(t)$, which for every $s \in (0, 1)$ is holomorphic in $|t| < (1-s)T_0$ with values in X_s , and is a solution to the Cauchy problem (2.2).

Next we restate the Cauchy problem (1.1) in a more convenient form. Note that (1.1) is equivalent to the following one.

$$u_t - u_{xxt} + 3u^2u_x - 4uu_xu_{xx} + u_x^2u_{xxx} + 2u_xu_{xx}^2 - u^2u_{xxx} - u_x^3 = 0.$$

Applying the operator $(1 - \partial_x^2)^{-1}$ to both sides of the above equation, we obtain

$$u_t + \left(u^2 - \frac{1}{3}u_x^2\right)u_x + \partial_x(1 - \partial_x^2)^{-1} \left(\frac{2}{3}u^3 + uu_x^2\right) + (1 - \partial_x^2)^{-1} \frac{u_x^3}{3} = 0.$$

Differentiating with respect to x on both sides of the above equation and noticing that $\partial_x^2(1 - \partial_x^2)^{-1} = (1 - \partial_x^2)^{-1} - I$, one finds that

$$u_{tx} + u^2u_{xx} + uu_x^2 - u_x^2u_{xx} - \frac{2}{3}u^3 + (1 - \partial_x^2)^{-1} \left(\frac{2}{3}u^3 + uu_x^2\right) + \partial_x(1 - \partial_x^2)^{-1} \frac{u_x^3}{3} = 0.$$

Let $v = u_x$. Then the problem (1.1) can be written as a system for u and v :

$$\begin{cases} u_t = -u^2v + \frac{1}{3}v^3 - \partial_x(1 - \partial_x^2)^{-1} \left(\frac{2}{3}u^3 + uv^2\right) - (1 - \partial_x^2)^{-1} \frac{v^3}{3}, \\ v_t = -uv^2 + \frac{2}{3}u^3 + v^2v_x - u^2v_x - (1 - \partial_x^2)^{-1} \left(\frac{2}{3}u^3 + uv^2\right) - \partial_x(1 - \partial_x^2)^{-1} \frac{v^3}{3}, \\ u(0, x) = u_0(x), \quad v(0, x) = u'_0(x). \end{cases}$$

Note that the initial data in the Cauchy problem (2.2) of the abstract Cauchy-Kowalevski theorem equals to zero, one can set $U = u - u_0$, $V = v - u'_0$, then the above problem is equivalent to

$$\begin{cases} U_t = -(U + u_0)^2(V + u'_0) - \partial_x(1 - \partial_x^2)^{-1} \left(\frac{2}{3}(U + u_0)^3 + (U + u_0)(V + u'_0)^2\right) \\ \quad - (1 - \partial_x^2)^{-1} \frac{1}{3}(V + u'_0)^3 + \frac{1}{3}(V + u'_0)^3 = F(U, V), \\ V_t = -(U + u_0)(V + u'_0)^2 + \frac{2}{3}(U + u_0)^3 + (V + u'_0)^2(V + u'_0)_x \\ \quad - (1 - \partial_x^2)^{-1} \left(\frac{2}{3}(U + u_0)^3 + (U + u_0)(V + u'_0)^2\right) \\ \quad - \partial_x(1 - \partial_x^2)^{-1} \frac{1}{3}(V + u'_0)^3 - (U + u_0)^2(V + u'_0)_x = G(U, V), \\ U(0, x) = 0, \quad V(0, x) = 0. \end{cases} \quad (2.3)$$

This can be written as the abstract form of the Cauchy problem in (2.2).

3. LOCAL WELL-POSEDNESS

In this section, we shall discuss the local well-posedness of the Cauchy problem (1.1). At first, we present the following definition.

Definition 3.1. For $T > 0$, $s \in \mathbb{R}$ and $1 \leq p \leq +\infty$, we set

$$\begin{aligned} E_{p,r}^s(T) &\triangleq \mathcal{C}([0, T]; B_{p,r}^s) \cap \mathcal{C}^1([0, T]; B_{p,r}^{s-1}) \quad \text{if } r < +\infty, \\ E_{p,\infty}^s(T) &\triangleq L^\infty([0, T]; B_{p,\infty}^s) \cap Lip([0, T]; B_{p,\infty}^{s-1}) \end{aligned}$$

and $E_{p,r}^s \triangleq \cap_{T>0} E_{p,r}^s(T)$.

Our main local existence result is the following theorem.

Theorem 3.1. *Suppose that $1 \leq p, r \leq +\infty$, $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$ and $u_0 \in B_{p,r}^s$. Then there exists a time $T > 0$ such that the initial-value problem (1.1) has a unique solution $u \in E_{p,r}^s(T)$, and the map $u_0 \mapsto u$ is continuous from a neighborhood of u_0 in $B_{p,r}^s$ into*

$$\mathcal{C}([0, T]; B_{p,r}^{s'}) \cap \mathcal{C}^1([0, T]; B_{p,r}^{s'-1})$$

for every $s' < s$ when $r = +\infty$ and $s' = s$ whereas $r < +\infty$.

Remark 3.1. When $p = r = 2$, the Besov space $B_{p,r}^s(\mathbb{R})$ coincides with the Sobolev space $H^s(\mathbb{R})$. Theorem 3.1 implies that under the condition $u_0 \in H^s(\mathbb{R})$ with $s > 5/2$, we can obtain the local well-posedness for the initial-value problem (1.1).

Remark 3.2. The existence time for the initial-value problem (1.1) may be chosen independently of s in the following sense [33]. If

$$u \in \mathcal{C}([0, T]; H^s) \cap \mathcal{C}^1([0, T]; H^{s-1})$$

is the solution of the initial-value problem (1.1) with initial data $u_0 \in H^r$ for some $r > 5/2$, $r \neq s$, then

$$u \in \mathcal{C}([0, T]; H^r) \cap \mathcal{C}^1([0, T]; H^{r-1})$$

with the same time T . In particular, if $u_0 \in H^\infty$, then $u \in C([0, T]; H^\infty)$.

In the following, we denote $C > 0$ a generic constant only depending on p, r, s . Uniqueness and continuity with respect to the initial data are an immediate consequence of the following result.

Proposition 3.1. *Let $1 \leq p, r \leq +\infty$ and $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$. Let $u^{(1)}, u^{(2)}$ be two given solutions of the initial-value problem (1.1) with the initial data $u_0^{(1)}, u_0^{(2)} \in B_{p,r}^s$ satisfying $u^{(1)}, u^{(2)} \in L^\infty([0, T]; B_{p,r}^s) \cap \mathcal{C}([0, T]; \mathcal{S}')$. Then for every $t \in [0, T]$:*

$$\begin{aligned} & \|u^{(1)}(t) - u^{(2)}(t)\|_{B_{p,r}^{s-1}} \\ & \leq \|u_0^{(1)} - u_0^{(2)}\|_{B_{p,r}^{s-1}} \exp \left\{ C \int_0^t (\|u^{(1)}(\tau)\|_{B_{p,r}^s}^2 + \|u^{(2)}(\tau)\|_{B_{p,r}^s}^2) d\tau \right\}. \end{aligned} \quad (3.1)$$

Proof. Denote $u^{(12)} \triangleq u^{(2)} - u^{(1)}$ and $m^{(12)} \triangleq m^{(2)} - m^{(1)}$. It is obvious that

$$u^{(12)} \in L^\infty([0, T]; B_{p,r}^s) \cap \mathcal{C}([0, T]; \mathcal{S}'),$$

which implies that $u^{(12)} \in \mathcal{C}([0, T]; B_{p,r}^{s-1})$ and $u^{(12)}, m^{(12)}$ solves the transport equation

$$\begin{cases} \partial_t m^{(12)} + [(u^{(1)})^2 - (u_x^{(1)})^2] \partial_x m^{(12)} = f(u^{(12)}, m^{(12)}, u^{(1)}, u^{(2)}, m^{(1)}, m^{(2)}), \\ m^{(12)}|_{t=0} = m_0^{(12)} \triangleq m_0^{(2)} - m_0^{(1)}, \end{cases}$$

with

$$\begin{aligned} f(u^{(12)}, m^{(12)}, u^{(1)}, u^{(2)}, m^{(1)}, m^{(2)}) := & -[(u^{(12)}(u^{(1)} + u^{(2)}) - u_x^{(12)}(u_x^{(1)} + u_x^{(2)}))m_x^{(2)} \\ & - 2u_x^{(1)}m^{(12)}(m^{(1)} + m^{(2)}) - 2u_x^{(12)}(m^{(2)})^2]. \end{aligned}$$

According to Lemma 2.2 , we have

$$\begin{aligned}
& e^{-C \int_0^t \|\partial_x [(u^{(1)})^2 - (u_x^{(1)})^2](\tau)\|_{B_{p,r}^{s-2}} d\tau} \|m^{(12)}(t)\|_{B_{p,r}^{s-3}} \\
& \leq \|m_0^{(12)}\|_{B_{p,r}^{s-3}} + C \int_0^t e^{-C \int_0^\tau \|\partial_x [(u^{(1)})^2 - (u_x^{(1)})^2](\tau')\|_{B_{p,r}^{s-2}} d\tau'} \\
& \quad \times \|f(u^{(12)}, m^{(12)}, u^{(1)}, u^{(2)}, m^{(1)}, m^{(2)})(\tau)\|_{B_{p,r}^{s-3}} d\tau.
\end{aligned} \tag{3.2}$$

For $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$, by Lemma 2.4 we have

$$\begin{aligned}
& \| [u^{(12)}(u^{(1)} + u^{(2)}) - u_x^{(12)}(u_x^{(1)} + u_x^{(2)})] m_x^{(2)} \|_{B_{p,r}^{s-3}} \\
& \leq C \|m^{(2)}\|_{B_{p,r}^{s-2}} \|u^{(12)}\|_{B_{p,r}^{s-1}} (\|u^{(1)}\|_{B_{p,r}^{s-1}} + \|u^{(2)}\|_{B_{p,r}^{s-1}}) \\
& \leq C \|u^{(12)}\|_{B_{p,r}^{s-1}} (\|u^{(1)}\|_{B_{p,r}^s}^2 + \|u^{(2)}\|_{B_{p,r}^s}^2), \\
& \|u_x^{(1)} m^{(12)}(m^{(1)} + m^{(2)})\|_{B_{p,r}^{s-3}} \\
& \leq C \|u^{(1)}\|_{B_{p,r}^{s-1}} \|m^{(12)}\|_{B_{p,r}^{s-3}} (\|m^{(1)}\|_{B_{p,r}^{s-2}} + \|m^{(2)}\|_{B_{p,r}^{s-2}}) \\
& \leq C \|u^{(12)}\|_{B_{p,r}^{s-1}} (\|u^{(1)}\|_{B_{p,r}^s}^2 + \|u^{(2)}\|_{B_{p,r}^s}^2),
\end{aligned}$$

and

$$\|u_x^{(12)}(m^{(2)})^2\|_{B_{p,r}^{s-3}} \leq C \|u^{(12)}\|_{B_{p,r}^{s-2}} \|m^{(2)}\|_{B_{p,r}^{s-2}}^2 \leq C \|u^{(12)}\|_{B_{p,r}^{s-1}} \|u^{(2)}\|_{B_{p,r}^s}^2.$$

Therefore, inserting the above estimates to (3.2) we obtain

$$\begin{aligned}
& e^{-C \int_0^t \|\partial_x [(u^{(1)})^2 - (u_x^{(1)})^2](\tau)\|_{B_{p,r}^{s-2}} d\tau} \|u^{(12)}(t)\|_{B_{p,r}^{s-1}} \\
& \leq \|u_0^{(12)}\|_{B_{p,r}^{s-1}} + C \int_0^t e^{-C \int_0^\tau \|\partial_x [(u^{(1)})^2 - (u_x^{(1)})^2](\tau')\|_{B_{p,r}^{s-2}} d\tau'} \\
& \quad \times \|u^{(12)}(\tau)\|_{B_{p,r}^{s-1}} (\|u^{(1)}\|_{B_{p,r}^s}^2 + \|u^{(2)}\|_{B_{p,r}^s}^2) d\tau.
\end{aligned}$$

Hence, thanks to

$$\|\partial_x [(u^{(1)})^2 - (u_x^{(1)})^2]\|_{B_{p,r}^{s-2}} \leq C (\|u^{(1)}\|_{B_{p,r}^s}^2 + \|u^{(2)}\|_{B_{p,r}^s}^2)$$

and then applying Gronwall's inequality, we reach (3.1). \square

Now let us start the proof of Theorem 3.1, which is motivated by the proof of local existence theorem about the Camassa-Holm equation in [12]. Firstly, we shall use the classical Friedrichs regularization method to construct the approximate solutions to the Cauchy problem (1.1).

Lemma 3.1. *Let u_0, p, r and s be as in the statement of Theorem 3.1. Assume that $u^{(0)} := 0$. There exists a sequence of smooth functions $(u^{(n)})_{n \in \mathbb{N}} \in \mathcal{C}(\mathbb{R}^+; B_{p,r}^\infty)$ solving the following linear transport equation by induction:*

$$(T_n) \quad \begin{cases} \{\partial_t + [(u^{(n)})^2 - (u_x^{(n)})^2] \partial_x\} m^{(n+1)} = -2u_x^{(n)}(m^{(n)})^2, & t > 0, x \in \mathbb{R}, \\ u^{(n+1)}|_{t=0} = u_0^{(n+1)}(x) = S_{n+1}u_0, & x \in \mathbb{R}. \end{cases} \tag{3.3}$$

Moreover, there exists a $T > 0$ such that the solutions satisfying the following properties:

- (i) $(u^{(n)})_{n \in \mathbb{N}}$ is uniformly bounded in $E_{p,r}^s(T)$.
- (ii) $(u^{(n)})_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0, T]; B_{p,r}^{s-1})$.

Proof. Since all data $S_{n+1}u_0$ belongs to $B_{p,r}^\infty$, Lemma 2.3 enables us to show by induction that for all $n \in \mathbb{N}$, the equation (T_n) has a global solution which belongs to $\mathcal{C}(\mathbb{R}; B_{p,r}^\infty)$. Thanks to Lemma 2.2 and the proof of Proposition 3.1, we have the following inequality for all $n \in \mathbb{N}$:

$$\begin{aligned} & e^{-C \int_0^t \|\partial_x [(u^{(n)})^2 - (u_x^{(n)})^2](\tau)\|_{B_{p,r}^{s-2}} d\tau} \|m^{(n+1)}(t)\|_{B_{p,r}^{s-2}} \\ & \leq \|m_0\|_{B_{p,r}^{s-2}} + C \int_0^t e^{-C \int_0^\tau \|\partial_x [(u^{(n)})^2 - (u_x^{(n)})^2](\tau')\|_{B_{p,r}^{s-2}} d\tau'} \|u_x^{(n)}(m^{(n)})^2\|_{B_{p,r}^{s-2}} d\tau. \end{aligned}$$

Thanks to $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$, we find $B_{p,r}^{s-2}$ is an algebra. From this, one obtains

$$\|u_x^{(n)}(m^{(n)})^2\|_{B_{p,r}^{s-2}} \leq C \|u_x^{(n)}\|_{B_{p,r}^{s-2}} \|m^{(n)}\|_{B_{p,r}^{s-2}}^2 \leq C \|u^{(n)}\|_{B_{p,r}^s}^3,$$

which along with the above inequality leads to

$$\begin{aligned} & e^{-C \int_0^t \|\partial_x [(u^{(n)})^2 - (u_x^{(n)})^2](\tau)\|_{B_{p,r}^{s-2}} d\tau} \|u^{(n+1)}(t)\|_{B_{p,r}^s} \\ & \leq \|u_0\|_{B_{p,r}^s} + C \int_0^t e^{-C \int_0^\tau \|\partial_x [(u^{(n)})^2 - (u_x^{(n)})^2](\tau')\|_{B_{p,r}^{s-2}} d\tau'} \|u^{(n)}(\tau)\|_{B_{p,r}^s}^3 d\tau. \end{aligned}$$

Hence, we get

$$\begin{aligned} \|u^{(n+1)}(t)\|_{B_{p,r}^s} & \leq e^{C \int_0^t \|\partial_x [(u^{(n)})^2 - (u_x^{(n)})^2](\tau)\|_{B_{p,r}^{s-2}} d\tau} \|u_0\|_{B_{p,r}^s} \\ & \quad + C \int_0^t e^{C \int_\tau^t \|\partial_x [(u^{(n)})^2 - (u_x^{(n)})^2](\tau')\|_{B_{p,r}^{s-2}} d\tau'} \|u^{(n)}(\tau)\|_{B_{p,r}^s}^3 d\tau. \end{aligned} \tag{3.4}$$

Let us choose a $T > 0$ such that $4C\|u_0\|_{B_{p,r}^s}^2 T < 1$, and suppose by induction that for all $t \in [0, T]$

$$\|u^{(n)}(t)\|_{B_{p,r}^s} \leq \frac{\|u_0\|_{B_{p,r}^s}}{(1 - 4C\|u_0\|_{B_{p,r}^s}^2 t)^{1/2}}. \tag{3.5}$$

Indeed, since $B_{p,r}^{s-2}$ is an algebra, one obtains from (3.5) that for any $0 \leq \tau \leq t$

$$\begin{aligned} & C \int_\tau^t \|\partial_x [(u^{(n)})^2 - (u_x^{(n)})^2](\tau')\|_{B_{p,r}^{s-2}} d\tau' \leq C \int_\tau^t \|u^{(n)}(\tau')\|_{B_{p,r}^s}^2 d\tau' \\ & \leq C \int_\tau^t \frac{\|u_0\|_{B_{p,r}^s}^2}{1 - 4C\|u_0\|_{B_{p,r}^s}^2 \tau'} d\tau' = \frac{1}{4} \ln(1 - 4C\|u_0\|_{B_{p,r}^s}^2 \tau) - \frac{1}{4} \ln(1 - 4C\|u_0\|_{B_{p,r}^s}^2 t). \end{aligned}$$

And then inserting the above inequality and (3.5) into (3.4) leads to

$$\begin{aligned} \|u^{(n+1)}(t)\|_{B_{p,r}^s} & \leq \frac{\|u_0\|_{B_{p,r}^s}}{\sqrt[4]{1 - 4C\|u_0\|_{B_{p,r}^s}^2 t}}} + \frac{C}{\sqrt[4]{1 - 4C\|u_0\|_{B_{p,r}^s}^2 t}}} \\ & \quad \times \int_0^t \sqrt[4]{1 - 4C\|u_0\|_{B_{p,r}^s}^2 \tau} \frac{\|u_0\|_{B_{p,r}^s}^3}{(1 - 4C\|u_0\|_{B_{p,r}^s}^2 \tau)^{\frac{3}{2}}} d\tau \\ & \leq \frac{\|u_0\|_{B_{p,r}^s}}{\sqrt[4]{1 - 4C\|u_0\|_{B_{p,r}^s}^2 t}}} \left(1 + C \int_0^t \frac{\|u_0\|_{B_{p,r}^s}^2}{(1 - 4C\|u_0\|_{B_{p,r}^s}^2 \tau)^{\frac{5}{4}}} d\tau \right), \end{aligned}$$

which implies

$$\begin{aligned}\|u^{(n+1)}(t)\|_{B_{p,r}^s} &\leq \frac{\|u_0\|_{B_{p,r}^s}}{\sqrt[4]{1-4C\|u_0\|_{B_{p,r}^s}^2 t}} \left(1 + \frac{1}{\sqrt[4]{1-4C\|u_0\|_{B_{p,r}^s}^2 t}} - 1\right) \\ &= \frac{\|u_0\|_{B_{p,r}^s}}{\sqrt{1-4C\|u_0\|_{B_{p,r}^s}^2 t}}.\end{aligned}$$

Hence, one can see that

$$\|u^{(n+1)}(t)\|_{B_{p,r}^s} \leq \frac{\|u_0\|_{B_{p,r}^s}}{(1-4C\|u_0\|_{B_{p,r}^s}^2 t)^{1/2}}.$$

Therefore, $(u^{(n)})_{n \in \mathbb{N}}$ is uniformly bounded in $\mathcal{C}([0, T]; B_{p,r}^s)$. Using the Moser-type estimates (see Lemma 2.4 (ii)), one finds that

$$\begin{aligned}\|[(u^{(n)})^2 - (u_x^{(n)})^2] \partial_x m^{(n+1)}\|_{B_{p,r}^{s-3}} &\leq C \|m^{(n+1)}\|_{B_{p,r}^{s-2}} (\|u^{(n)}\|_{B_{p,r}^s}^2 + \|u_x^{(n)}\|_{B_{p,r}^{s-1}}^2) \\ &\leq C \|u^{(n+1)}\|_{B_{p,r}^s} \|u^{(n)}\|_{B_{p,r}^s}^2,\end{aligned}$$

and

$$\|u_x^{(n)} (m^{(n)})^2\|_{B_{p,r}^{s-3}} \leq C \|m^{(n)}\|_{B_{p,r}^{s-2}}^2 \|u^{(n)}\|_{B_{p,r}^s} \leq C \|u^{(n)}\|_{B_{p,r}^s}^3.$$

Hence, using the equation (T_n) , we have

$$\partial_t u^{(n+1)} \in \mathcal{C}([0, T]; B_{p,r}^{s-1})$$

uniformly bounded, which yields that the sequence $(u^{(n)})_{n \in \mathbb{N}}$ is uniformly bounded in $E_{p,r}^s(T)$.

Next we are going to show that

$$(u^{(n)})_{n \in \mathbb{N}} \text{ is a Cauchy sequence in } \mathcal{C}([0, T]; B_{p,r}^{s-1}).$$

In fact, according to (3.3), we obtain that, for all $n, l \in \mathbb{N}$,

$$\begin{aligned}&\{\partial_t + [(u^{(n+l)})^2 - (u_x^{(n+l)})^2] \partial_x\} (m^{(n+l+1)} - m^{(n+1)}) \\ &= g(u^{(n+l)}, u^{(n)}, m^{(n+l)}, m^{(n)}, m^{(n+1)}),\end{aligned}$$

where

$$\begin{aligned}&g(u^{(n+l)}, u^{(n)}, m^{(n+l)}, m^{(n)}, m^{(n+1)}) \\ &= [(u^{(n)} - u^{(n+l)})(u^{(n)} + u^{(n+l)}) - (u_x^{(n)} - u_x^{(n+l)})(u_x^{(n)} + u_x^{(n+l)})] \partial_x m^{(n+1)} \\ &\quad - 2u_x^{(n+l)} (m^{(n+l)} - m^{(n)}) (m^{(n)} + m^{(n+l)}) + 2(u_x^{(n)} - u_x^{(n+l)})(m^{(n)})^2.\end{aligned}$$

Applying Lemma 2.2 again, then for every $t \in [0, T]$, we obtain

$$\begin{aligned}&e^{-C \int_0^t \|[(u^{(n+l)})^2 - (u_x^{(n+l)})^2](\tau)\|_{B_{p,r}^{s-1}} d\tau} \|(m^{(n+l+1)} - m^{(n+1)})(t)\|_{B_{p,r}^{s-3}} \\ &\leq \|m_0^{(n+l+1)} - m_0^{(n+1)}\|_{B_{p,r}^{s-3}} + C \int_0^t e^{-C \int_0^\tau \|[(u^{(n+l)})^2 - (u_x^{(n+l)})^2](\tau')\|_{B_{p,r}^{s-1}} d\tau'} \\ &\quad \times \|g(u^{(n+l)}, u^{(n)}, m^{(n+l)}, m^{(n)}, m^{(n+1)})\|_{B_{p,r}^{s-3}} d\tau,\end{aligned}$$

which is equivalent to

$$\begin{aligned}
& e^{-C \int_0^t \|[(u^{(n+l)})^2 - (u_x^{(n+l)})^2](\tau)\|_{B_{p,r}^{s-1}} d\tau} \|(u^{(n+l+1)} - u^{(n+1)})(t)\|_{B_{p,r}^{s-1}} \\
& \leq \|u_0^{(n+l+1)} - u_0^{(n+1)}\|_{B_{p,r}^{s-1}} + C \int_0^t e^{-C \int_0^\tau \|[(u^{(n+l)})^2 - (u_x^{(n+l)})^2](\tau')\|_{B_{p,r}^{s-1}} d\tau'} \\
& \quad \times \|g(u^{(n+l)}, u^{(n)}, m^{(n+l)}, m^{(n)}, m^{(n+1)})\|_{B_{p,r}^{s-3}} d\tau.
\end{aligned}$$

In the case of $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$, one can deduce that

$$\begin{aligned}
& \|[(u^{(n)} - u^{(n+l)})(u^{(n)} + u^{(n+l)}) - (u_x^{(n)} - u_x^{(n+l)})(u_x^{(n)} + u_x^{(n+l)})]\partial_x m^{(n+1)}\|_{B_{p,r}^{s-3}} \\
& \leq C \|m^{(n+1)}\|_{B_{p,r}^{s-2}} (\|u^{(n+l)} - u^{(n)}\|_{B_{p,r}^{s-1}} \|u^{(n+l)} + u^{(n)}\|_{B_{p,r}^{s-1}} \\
& \quad + \|u_x^{(n+l)} - u_x^{(n)}\|_{B_{p,r}^{s-2}} \|u_x^{(n+l)} - u_x^{(n)}\|_{B_{p,r}^{s-2}}) \\
& \leq C \|u^{(n+l)} - u^{(n)}\|_{B_{p,r}^{s-1}} (\|u^{(n)}\|_{B_{p,r}^s}^2 + \|u^{(n+1)}\|_{B_{p,r}^s}^2 + \|u^{(n+l)}\|_{B_{p,r}^s}^2), \\
& \quad \|u_x^{(n+l)}(m^{(n+l)} - m^{(n)})(m^{(n)} + m^{(n+l)})\|_{B_{p,r}^{s-3}} \\
& \leq C \|u^{(n+l)}\|_{B_{p,r}^s} \|m^{(n+l)} - m^{(n)}\|_{B_{p,r}^{s-3}} \|m^{(n+l)} + m^{(n)}\|_{B_{p,r}^{s-2}} \\
& \leq C \|u^{(n+l)} - u^{(n)}\|_{B_{p,r}^{s-1}} (\|u^{(n)}\|_{B_{p,r}^s}^2 + \|u^{(n+1)}\|_{B_{p,r}^s}^2 + \|u^{(n+l)}\|_{B_{p,r}^s}^2),
\end{aligned}$$

and

$$\begin{aligned}
& \|(u_x^{(n)} - u_x^{(n+l)})(m^{(n)})^2\|_{B_{p,r}^{s-3}} \leq C \|u^{(n+l)} - u^{(n)}\|_{B_{p,r}^{s-2}} \|m^{(n)}\|_{B_{p,r}^{s-2}}^2 \\
& \leq C \|u^{(n+l)} - u^{(n)}\|_{B_{p,r}^{s-1}} \|u^{(n)}\|_{B_{p,r}^s}^2.
\end{aligned}$$

From this, one finds that

$$\begin{aligned}
& \|g(u^{(n+l)}, u^{(n)}, m^{(n+l)}, m^{(n)}, m^{(n+1)})\|_{B_{p,r}^{s-3}} \\
& \leq C \|u^{(n+l)} - u^{(n)}\|_{B_{p,r}^{s-1}} (\|u^{(n)}\|_{B_{p,r}^s}^2 + \|u^{(n+1)}\|_{B_{p,r}^s}^2 + \|u^{(n+l)}\|_{B_{p,r}^s}^2).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
& e^{-C \int_0^t \|[(u^{(n+l)})^2 - (u_x^{(n+l)})^2](\tau)\|_{B_{p,r}^{s-1}} d\tau} \|(u^{(n+l+1)} - u^{(n+1)})(t)\|_{B_{p,r}^{s-1}} \\
& \leq \|u_0^{(n+l+1)} - u_0^{(n+1)}\|_{B_{p,r}^{s-1}} + C \int_0^t e^{-C \int_0^\tau \|[(u^{(n+l)})^2 - (u_x^{(n+l)})^2](\tau')\|_{B_{p,r}^{s-1}} d\tau'} \\
& \quad \times \|(u^{(n+l)} - u^{(n)})(\tau)\|_{B_{p,r}^{s-1}} (\|u^{(n)}(\tau)\|_{B_{p,r}^s}^2 + \|u^{(n+1)}(\tau)\|_{B_{p,r}^s}^2 + \|u^{(n+l)}(\tau)\|_{B_{p,r}^s}^2) d\tau.
\end{aligned}$$

Since $(u^{(n)})_{n \in \mathbb{N}}$ is uniformly bounded in $E_{p,r}^s(T)$ and

$$u_0^{(n+l+1)} - u_0^{(n+1)} = S_{n+l+1}u_0 - S_{n+1}u_0 = \sum_{q=n+1}^{n+l} \Delta_q u_0,$$

then there exists a constant C_T independent of n and l such that for all $t \in [0, T]$

$$\|(u^{(n+l+1)} - u^{(n+1)})(t)\|_{B_{p,r}^{s-1}} \leq C_T \left(2^{-n} + \int_0^t \|(u^{(n+l)} - u^{(n)})(\tau)\|_{B_{p,r}^{s-1}} d\tau \right).$$

Arguing by induction with respect to the index n , one can easily prove that

$$\|u^{(n+l+1)} - u^{(n+1)}\|_{L_T^\infty(B_{p,r}^{s-1})} \leq \frac{(TC_T)^{n+1}}{(n+1)!} \|u^{(l)}\|_{L_T^\infty(B_{p,r}^s)} + C_T \sum_{k=0}^n 2^{-(n-k)} \frac{(TC_T)^k}{k!}.$$

Similarly $\|u^{(l)}\|_{L_T^\infty(B_{p,r}^s)}$ can be bounded independently of l , we conclude that there exist some new constant C'_T independent of n and l such that

$$\|u^{(n+l+1)} - u^{(n+1)}\|_{L_T^\infty(B_{p,r}^{s-1})} \leq C'_T 2^{-n}.$$

Hence $(u^{(n)})_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0, T]; B_{p,r}^{s-1})$. \square

Proof of Theorem 3.1. Thanks to Lemma 3.1, we obtain that $(u^{(n)})_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0, T]; B_{p,r}^{s-1})$, so it converges to some function $u \in \mathcal{C}([0, T]; B_{p,r}^{s-1})$. We now have to check that u belongs to $E_{p,r}^s(T)$ and solves the Cauchy problem (1.1). Since $(u^{(n)})_{n \in \mathbb{N}}$ is uniformly bounded in $L^\infty([0, T]; B_{p,r}^s)$ according to Lemma 3.1, the Fatou property for the Besov spaces (Proposition 2.2 iv)) guarantees that u also belongs to $L^\infty([0, T]; B_{p,r}^s)$.

On the other hand, as $(u^{(n)})_{n \in \mathbb{N}}$ converges to u in $\mathcal{C}([0, T]; B_{p,r}^{s-1})$, an interpolation argument ensures that the convergence holds in $\mathcal{C}([0, T]; B_{p,r}^{s'})$, for any $s' < s$. It is then easy to pass to the limit in the equation (T_n) and to conclude that u is indeed a solution to the Cauchy problem (1.1). Thanks to the fact that u belongs to $L^\infty([0, T]; B_{p,r}^s)$, the right-hand side of the equation

$$\partial_t m + (u^2 - u_x^2) \partial_x m = -2u_x m^2$$

belongs to $L^\infty([0, T]; B_{p,r}^{s-2})$. In particular, for the case $r < \infty$, Lemma 2.3 enables us to conclude that $u \in \mathcal{C}([0, T]; B_{p,r}^{s'})$ for any $s' < s$. Finally, using the equation again, we see that $\partial_t u \in \mathcal{C}([0, T]; B_{p,r}^{s-1})$ if $r < \infty$, and in $L^\infty([0, T]; B_{p,r}^{s-1})$ otherwise. Moreover, a standard use of a sequence of viscosity approximate solutions $(u_\epsilon)_{\epsilon > 0}$ for the Cauchy problem (1.1) which converges uniformly in

$$\mathcal{C}([0, T]; B_{p,r}^s) \cap \mathcal{C}^1([0, T]; B_{p,r}^{s-1})$$

gives the continuity of the solution u in $E_{p,r}^s(T)$. \square

4. BLOW-UP SCENARIO AND GLOBAL CONSERVATIVE PROPERTY

In this section, attention is now turned to blow-up issue. We first present a blow-up scenario.

Theorem 4.1. *Let $u_0 \in H^s$, $s > 5/2$, and $u(t, x)$ be the solution of the Cauchy problem (1.1) with life-span T . Then T is finite if and only if*

$$\liminf_{t \uparrow T} \left[\inf_{x \in \mathbb{R}} (mu_x(t, x)) \right] = -\infty.$$

Proof. Since the existence time T is independent of the choice of s , in view of Remark 3.2, we only need to consider the case $s = 3$ by utilizing a simple density argument. Multiplying Eq.(1.1) by m and integrating over \mathbb{R} with respect to x yield

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} m^2 dx &= - \int_{\mathbb{R}} (u^2 - u_x^2) m m_x dx - 2 \int_{\mathbb{R}} u_x m^3 dx \\ &= \frac{1}{2} \int_{\mathbb{R}} (u^2 - u_x^2)_x m^2 dx - 2 \int_{\mathbb{R}} u_x m^3 dx \\ &= - \int_{\mathbb{R}} u_x m^3 dx. \end{aligned}$$

Differentiating the first equation with regard to x , one finds that

$$\begin{aligned} m_{xt} &= -2u_{xx}m^2 - 6u_xmm_x - (u^2 - u_x^2)m_{xx} \\ &= -2um^2 + 2m^3 - 6u_xmm_x - (u^2 - u_x^2)m_{xx}. \end{aligned}$$

Then multiplying the above equation by m_x and integrating over \mathbb{R} with respect to x , it leads to

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} m_x^2 dx \\ &= - \int_{\mathbb{R}} (u^2 - u_x^2)m_x m_{xx} dx - 2 \int_{\mathbb{R}} um^2 m_x dx - 6 \int_{\mathbb{R}} u_x mm_x^2 dx + 2 \int_{\mathbb{R}} m^3 m_x dx \\ &= \frac{1}{2} \int_{\mathbb{R}} (u^2 - u_x^2)_x m_x^2 dx + \frac{2}{3} \int_{\mathbb{R}} u_x m^3 dx - 6 \int_{\mathbb{R}} u_x mm_x^2 dx \\ &= -5 \int_{\mathbb{R}} u_x mm_x^2 dx + \frac{2}{3} \int_{\mathbb{R}} u_x m^3 dx. \end{aligned}$$

Therefore,

$$\frac{d}{dt} \int_{\mathbb{R}} (m^2 + m_x^2) dx = -10 \int_{\mathbb{R}} u_x mm_x^2 dx - \frac{2}{3} \int_{\mathbb{R}} u_x m^3 dx.$$

If mu_x is bounded from below on $[0, T) \times \mathbb{R}$, i.e., there exists $N > 0$ such that $mu_x \geq -N$ on $[0, T) \times \mathbb{R}$, then it is thereby inferred from the above estimate that

$$\frac{d}{dt} \int_{\mathbb{R}} (m^2 + m_x^2) dx \leq 10N \int_{\mathbb{R}} (m^2 + m_x^2) dx.$$

Applying Gronwall inequality then yields for $t \in [0, T)$

$$\|m\|_{H^1}^2 \leq \int_{\mathbb{R}} (m^2 + m_x^2) dx \leq e^{10NT} \int_{\mathbb{R}} (m_0^2 + m_{0x}^2) dx = e^{10NT} \|m_0\|_{H^1}^2. \quad (4.1)$$

Differentiating the first equation with regard to x twice, one finds that

$$\begin{aligned} m_{xxt} &= -2u_xm^2 - 4umm_x + 6m^2m_x - 6u_xm_x^2 - 6u_xmm_{xx} - 6u_{xx}mm_x \\ &\quad - 2u_xmm_{xx} - (u^2 - u_x^2)m_{xxx} \\ &= -2u_xm^2 - 10umm_x + 12m^2m_x - 6u_xm_x^2 - 8u_xmm_{xx} - (u^2 - u_x^2)m_{xxx}. \end{aligned}$$

Then multiplying the above equation by m_{xx} and integrating over \mathbb{R} with respect to x , we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} m_{xx}^2 dx \\ &= - \int_{\mathbb{R}} (u^2 - u_x^2)m_{xxx}m_{xx} dx - 8 \int_{\mathbb{R}} u_x mm_{xx}^2 dx - 2 \int_{\mathbb{R}} u_x m^2 m_{xx} dx \\ &\quad + 12 \int_{\mathbb{R}} m^2 m_x m_{xx} dx - 10 \int_{\mathbb{R}} umm_x m_{xx} dx - 6 \int_{\mathbb{R}} u_x m_{xx} m_x^2 dx. \end{aligned}$$

For the right hand side of the above equation, integrating by parts one finds that

$$- \int_{\mathbb{R}} (u^2 - u_x^2)m_{xxx}m_{xx} dx = \frac{1}{2} \int_{\mathbb{R}} (u^2 - u_x^2)_x m_{xx}^2 dx = \int_{\mathbb{R}} u_x mm_{xx}^2 dx,$$

$$\begin{aligned}
-2 \int_{\mathbb{R}} u_x m^2 m_{xx} dx &= 2 \int_{\mathbb{R}} m_x (u_{xx} m^2 + 2u_x m m_x) dx \\
&= 4 \int_{\mathbb{R}} u_x m m_x^2 dx + 2 \int_{\mathbb{R}} (u - m) m^2 m_x dx \\
&= 4 \int_{\mathbb{R}} u_x m m_x^2 dx - \frac{2}{3} \int_{\mathbb{R}} u_x m^3 dx,
\end{aligned}$$

and

$$\begin{aligned}
-6 \int_{\mathbb{R}} u_x m_{xx} m_x^2 dx &= 2 \int_{\mathbb{R}} u_{xx} m_x^3 dx = 2 \int_{\mathbb{R}} u m_x^3 dx - 2 \int_{\mathbb{R}} m m_x^3 dx \\
&= -2 \int_{\mathbb{R}} u_x m m_x^2 dx - 4 \int_{\mathbb{R}} u m m_x m_{xx} dx \\
&\quad + 2 \int_{\mathbb{R}} m^2 m_x m_{xx} dx,
\end{aligned}$$

Where we have used

$$2 \int_{\mathbb{R}} u m_x^3 dx = -2 \int_{\mathbb{R}} m (u_x m_x^2 + 2u m_x m_{xx}) dx,$$

and

$$-2 \int_{\mathbb{R}} m m_x^3 dx = 2 \int_{\mathbb{R}} m (m_x^3 + 2m m_x m_{xx}) dx.$$

Therefore,

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} m_{xx}^2 dx &= -7 \int_{\mathbb{R}} u_x m m_{xx}^2 dx + 2 \int_{\mathbb{R}} u_x m m_x^2 dx - \frac{2}{3} \int_{\mathbb{R}} u_x m^3 dx \\
&\quad + 14 \int_{\mathbb{R}} m^2 m_x m_{xx} dx - 14 \int_{\mathbb{R}} u m m_x m_{xx} dx.
\end{aligned}$$

And so

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}} (m^2 + m_x^2 + m_{xx}^2) dx \\
&= -14 \int_{\mathbb{R}} u_x m m_{xx}^2 dx - 6 \int_{\mathbb{R}} u_x m m_x^2 dx - 2 \int_{\mathbb{R}} u_x m^3 dx \\
&\quad + 28 \int_{\mathbb{R}} m^2 m_x m_{xx} dx - 28 \int_{\mathbb{R}} u m m_x m_{xx} dx.
\end{aligned}$$

If mu_x is bounded from below on $[0, T) \times \mathbb{R}$, i.e., there exists $N > 0$ such that $mu_x \geq -N$ on $[0, T) \times \mathbb{R}$, then applying (4.1) we can deduce that

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}} (m^2 + m_x^2 + m_{xx}^2) dx \\
&\leq 14N \int_{\mathbb{R}} (m^2 + m_x^2 + m_{xx}^2) dx + 28(\|m\|_{L^\infty}^2 + \|um\|_{L^\infty}) \int_{\mathbb{R}} |m_x m_{xx}| dx \\
&\leq 14N \int_{\mathbb{R}} (m^2 + m_x^2 + m_{xx}^2) dx + 28\|m\|_{H^1}^2 \int_{\mathbb{R}} (m^2 + m_x^2 + m_{xx}^2) dx \\
&\leq 14(N + 2e^{10NT}\|m_0\|_{H^1}^2) \int_{\mathbb{R}} (m^2 + m_x^2 + m_{xx}^2) dx.
\end{aligned}$$

For any $t \in [0, T)$, using Gronwall inequality again it leads to

$$\begin{aligned}\|u\|_{H^4}^2 &\leq \|m\|_{H^2}^2 = \int_{\mathbb{R}} (m^2 + m_x^2 + m_{xx}^2) dx \\ &\leq \exp(14T(N + 2e^{10NT}\|m_0\|_{H^1}^2)) \int_{\mathbb{R}} (m_0^2 + m_{0x}^2 + m_{0xx}^2) dx \\ &= \exp(14T(N + 2e^{10NT}\|m_0\|_{H^1}^2)) \|u_0\|_{H^4}^2.\end{aligned}$$

The above inequality and Sobolev's embedding theorem ensure that the solution $u(t, x)$ does not blow up in finite time.

On the other hand, if

$$\liminf_{t \uparrow T} \left[\inf_{x \in \mathbb{R}} (mu_x(t, x)) \right] = -\infty,$$

by the existence Theorem 3.1 of the local strong solution and Sobolev's embedding theorem, we infer that the solution will blow-up in finite time. The proof of Theorem 4.1 is thus complete. \square

In order to demonstrate a conservative property, let us consider the trajectory equation

$$\begin{cases} \frac{dq}{dt} = (u^2 - u_x^2)(t, q(t, x)) \\ q(0, x) = x. \end{cases} \quad (4.2)$$

For all $t > 0$, a simple computation shows that

$$\begin{aligned}&\frac{\partial}{\partial t} [m(t, q(t, x))q_x(t, x)] \\ &= [m_t(t, q) + m_x(t, q)q_t]q_x + mq_{xt} \\ &= q_x[m_t(t, q) + (u^2 - u_x^2)m_x(t, q)] + 2u_x m^2 q_x \\ &= q_x[m_t + (u^2 - u_x^2)m_x + 2u_x m^2] = 0.\end{aligned}$$

Therefore, $m(t, q(t, x))q_x(t, x)$ is independent of the time variable t . That is

$$m(t, q(t, x))q_x(t, x) = m(0, x) = u_0(x) - u_{0xx}(x).$$

5. ANALYTICITY OF SOLUTIONS

In this section, we shall establish the following analyticity result.

Theorem 5.1. *Let u_0 be a real analytic function on \mathbb{R} . There exist an $\varepsilon > 0$ and a unique solution u of the Cauchy problem (1.1) that is analytic on $(-\varepsilon, \varepsilon) \times \mathbb{R}$.*

Note that it is sufficient to verify the conditions 1) and 2) in the statement of the abstract Cauchy-Kowalevski Theorem 2.1 for both $F(U, V)$ and $G(U, V)$ in the system (2.3) since neither F nor G depends on t explicitly. Note that u_0 is analytic by the assumption of Theorem 5.1, we can deduce that both $\|u_0\|_s$ and $\|u'_0\|_{s'}$ are bounded. Without loss of generality, we assume that there exist constants $M_0, M_1 > 0$ such that $\|u_0\|_s \leq M_0$, $\|u'_0\|_s \leq M_1$, and so $\|u'_0\|_{s'} \leq cM_0/(s - s')$. In order to prove 1), for $0 < s' < s < 1$, the estimates in Lemma 2.5 and 2.6 imply

the following bounds

$$\begin{aligned}
|||F(U, V)|||_{s'} &\leq c^2 |||U + u_0|||_s^2 |||V + u'_0|||_{s'} + \frac{2c^2}{3} |||V + u'_0|||_{s'}^3 + \frac{2c^2}{3} |||U + u_0|||_s^3 \\
&\quad + c^2 |||U + u_0|||_s |||V + u'_0|||_{s'}^2 \\
&\leq c^2 [(R + M_0)^2 (R + \frac{cM_0}{s - s'}) + \frac{2}{3} (R + \frac{cM_0}{s - s'})^3 \\
&\quad + \frac{2}{3} (R + M_0)^3 + (R + M_0)(R + \frac{cM_0}{s - s'})^2] \\
&\leq \frac{2c^2}{3} (2R + M_0 + \frac{cM_0}{s - s'})^3, \\
|||G(U, V)|||_{s'} &\leq 2c^2 |||U + u_0|||_s |||V + u'_0|||_{s'}^2 + \frac{4c^2}{3} |||U + u_0|||_s^3 + \frac{c^2}{3} |||V + u'_0|||_{s'}^3 \\
&\quad + \frac{c^3}{s - s'} |||U + u_0|||_s^2 |||V + u'_0|||_s + \frac{c^3}{s - s'} |||V + u'_0|||_s |||V + u'_0|||_{s'}^2 \\
&\leq 2c^2 (R + M_0)(R + \frac{cM_0}{s - s'})^2 + \frac{4c^2}{3} (R + M_0)^3 + \frac{c^2}{3} (R + \frac{cM_0}{s - s'})^3 \\
&\quad + \frac{c^3}{s - s'} (R + M_0)^2 (R + M_1) + \frac{c^3}{s - s'} (R + M_1)(R + \frac{cM_0}{s - s'})^2,
\end{aligned}$$

hence condition 1) holds.

Note that to verify the second condition it suffices to estimate

$$|||F(U_1, V) - F(U_2, V)|||_{s'}, |||F(U, V_1) - F(U, V_2)|||_{s'},$$

and

$$|||G(U_1, V) - G(U_2, V)|||_{s'}, |||G(U, V_1) - G(U, V_2)|||_{s'}.$$

Since

$$|||F(U_1, V_1) - F(U_2, V_2)|||_{s'} \leq |||F(U_1, V_1) - F(U_1, V_2)|||_{s'} + |||F(U_1, V_2) - F(U_2, V_2)|||_{s'},$$

and

$$|||G(U_1, V_1) - G(U_2, V_2)|||_{s'} \leq |||G(U_1, V_1) - G(U_1, V_2)|||_{s'} + |||G(U_1, V_2) - G(U_2, V_2)|||_{s'}.$$

Using this together with Lemma 2.5 and 2.6, we get the following estimates

$$\begin{aligned}
&|||F(U_1, V) - F(U_2, V)|||_{s'} \\
&\leq |||(V + u'_0)(U_1 - U_2)(U_1 + U_2 + 2u_0)|||_{s'} + |||\partial_x(1 - \partial_x^2)^{-1} [(U_1 - U_2)(V + u'_0)^2]|||_{s'} \\
&\quad + \frac{2}{3} |||\partial_x(1 - \partial_x^2)^{-1} \{(U_1 - U_2)[(U_1 + u_0)^2 - (U_1 + u_0)(U_2 + u_0) + (U_2 + u_0)^2]\}| |||_{s'} \\
&\leq 2c^2 (|||V + u'_0|||_{s'}^2 + |||U_1 + u_0|||_s^2 + |||U_2 + u_0|||_s^2) |||U_1 - U_2|||_s \\
&\leq 2c^2 [2(R + M_0)^2 + (R + \frac{cM_0}{s - s'})^2] |||U_1 - U_2|||_s,
\end{aligned}$$

$$\begin{aligned}
& |||F(U, V_1) - F(U, V_2)|||_{s'} \\
& \leq |||(U + u_0)^2(V_1 - V_2)|||_{s'} + \frac{1}{3} |||(V_1 - V_2)[(V_1 + u'_0)^2 - (V_1 + u'_0)(V_2 + u'_0) + (V_2 + u'_0)^2]|||_{s'} \\
& \quad + |||\partial_x(1 - \partial_x^2)^{-1}[(V_1 - V_2)(U + u_0)(V_1 + V_2 + 2u'_0)]|||_{s'} \\
& \quad + \frac{1}{3} |||(1 - \partial_x^2)^{-1}\{(V_1 - V_2)[(V_1 + u'_0)^2 - (V_1 + u'_0)(V_2 + u'_0) + (V_2 + u'_0)^2]\}|||_{s'} \\
& \leq 2c^2(|||V_1 + u'_0|||_{s'}^2 + |||V_2 + u'_0|||_{s'}^2 + |||U + u_0|||_s^2)|||V_1 - V_2|||_s \\
& \leq 2c^2[2(R + \frac{cM_0}{s - s'})^2 + (R + M_0)^2]|||V_1 - V_2|||_s,
\end{aligned}$$

$$\begin{aligned}
& |||G(U_1, V) - G(U_2, V)|||_{s'} \\
& \leq |||(V + u'_0)^2(U_1 - U_2)|||_{s'} + \frac{2}{3} |||(U_1 - U_2)[(U_1 + u_0)^2 - (U_1 + u_0)(U_2 + u_0) + (U_2 + u_0)^2]|||_{s'} \\
& \quad + |||(V_x + u''_0)(U_1 - U_2)(U_1 + U_2 + 2u_0)|||_{s'} + |||(1 - \partial_x^2)^{-1}[(V + u'_0)^2(U_1 - U_2)]|||_{s'} \\
& \quad + \frac{2}{3} |||(1 - \partial_x^2)^{-1}\{(U_1 - U_2)[(U_1 + u_0)^2 - (U_1 + u_0)(U_2 + u_0) + (U_2 + u_0)^2]\}|||_{s'} \\
& \leq (3c^2 + \frac{c^3}{2(s - s')^2})(|||V + u'_0|||_{s'}^2 + |||V + u'_0|||_s^2 + |||U_1 + u_0|||_s^2 + |||U_2 + u_0|||_s^2)|||U_1 - U_2|||_s \\
& \leq (3c^2 + \frac{c^3}{2(s - s')^2})[2(R + M_0)^2 + (R + M_1)^2 + (R + \frac{cM_0}{s - s'})^2]|||U_1 - U_2|||_s,
\end{aligned}$$

and

$$\begin{aligned}
& |||G(U, V_1) - G(U, V_2)|||_{s'} \\
& \leq |||(V_1 - V_2)(U + u_0)(V_1 + V_2 + 2u'_0)|||_{s'} + |||(V_1 - V_2)(V_1 + u'_0)_x(V_1 + V_2 + 2u'_0)|||_{s'} \\
& \quad + |||(1 - \partial_x^2)^{-1}[(V_1 - V_2)(U + u_0)(V_1 + V_2 + 2u'_0)]|||_{s'} + |||(V_2 + u'_0)^2(V_1 - V_2)_x|||_{s'} \\
& \quad + \frac{1}{3} |||\partial_x(1 - \partial_x^2)^{-1}\{(V_1 - V_2)[(V_1 + u'_0)^2 - (V_1 + u'_0)(V_2 + u'_0) + (V_2 + u'_0)^2]\}|||_{s'} \\
& \quad + |||(V_1 - V_2)_x(U + u_0)^2|||_{s'} \\
& \leq (4c^2 + \frac{c^3}{2(s - s')^2})(|||V_1 + u'_0|||_s^2 + |||V_1 + u'_0|||_{s'}^2 + |||V_2 + u'_0|||_{s'}^2 + |||U + u_0|||_s^2)|||V_1 - V_2|||_s \\
& \leq (4c^2 + \frac{c^3}{2(s - s')^2})[(R + M_0)^2 + (R + M_1)^2 + 2(R + \frac{cM_0}{s - s'})^2]|||V_1 - V_2|||_s.
\end{aligned}$$

This implies that the condition 2) also holds. Hence, the proof of Theorem 5.1 is complete.

6. APPENDIX A. GEOMETRIC DESCRIPTIONS

It has been known for long time that integrable equations solved by the inverse scattering transform method have elegant geometric interpretations. Several different geometric frameworks have been utilized to provide geometric interpretations to integrable systems. For instance the celebrated CH equation was shown to describe the geodesic flow of the Riemannian metric on the diffeomorphism group of the circle [21] and pseudo-spherical surface [29]. It also arises from a non-stretching invariant planar curve motion in the centro-equiaffine geometry [5]. What is more, the mKdV equation, the sine-Gordon equation, the Schrödinger equation, the KdV

equation and the Sawada-Kotera equation arise naturally from non-stretching invariant curve flows in Klein geometries (see [5, 6, 15, 16] and references therein).

In this appendix, we show that the equation (1.1) arises from non-stretching invariant curve flows respectively in two-dimensional Euclidean geometry and two-dimensional sphere, and it also describes a pseudo-spherical surface.

First, we study non-stretching invariant plane curve flows in the Euclidean geometry \mathbb{R}^2 , governed by

$$\frac{\partial \gamma}{\partial t} = f \mathbf{n} + g \mathbf{t}, \quad (\text{A.1})$$

where \mathbf{t} and \mathbf{n} are the Euclidean tangent and normal vectors, f and g are respectively the normal and tangent velocities depending on the curvature and its derivative with respect to the arc-length s of the curve. Let $ds = h dp$, where p is the free parameter independent of time t and h is the metric of the curve. A simple computation gives

$$h_t = (g_s - \kappa f)h.$$

Assume that the distance (along the curve) between any two points of the curve is invariant under the curve motion (A.1), that means $[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}] = 0$. Hence we deduce that

$$g_s - \kappa f = 0.$$

Let L be the parameter for a closed curve. A direct computation shows

$$\frac{\partial L}{\partial t} = \oint_{\gamma} (g_s - \kappa f) ds = - \oint_{\gamma} \kappa f ds.$$

Furthermore, assume that L is invariant under the curve flow (A.1). Then we require

$$\oint_{\gamma} \kappa f ds = 0.$$

By the curve flow (A.1), a straightforward computation leads to the evolution of the frame given by

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix}_t = \begin{pmatrix} 0 & f_s + \kappa g \\ -(f_s + \kappa g) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix}. \quad (\text{A.2})$$

Let θ be the angle between tangent vector of the curve and a fixed direction. Then $\mathbf{t} = (\cos \theta, \sin \theta)$, $\mathbf{n} = (-\sin \theta, \cos \theta)$. From (A.2), we get

$$\theta_t = f_s + \kappa g.$$

Hence the curvature $\kappa = \frac{d\theta}{ds}$ satisfies [15]

$$\kappa_t = (f_s + \kappa g)_s = \Omega f, \quad (\text{A.3})$$

where $\Omega = \partial_s^2 + \kappa^2 + \kappa_s \partial_s^{-1} \kappa$ is the recursion operator of the mKdV equation

$$\kappa_t = \kappa_{sss} + \frac{3}{2} \kappa^2 \kappa_s.$$

Set $f = -2v_s$, $\kappa = m \equiv v - v_{ss}$. Then $g = -(v^2 - v_s^2) + b$, b is a constant. Hence $v(t, s)$ satisfies the equation

$$m_t + [(v^2 - v_s^2)m]_s + (b + 2)v_{sss} - bv_s = 0. \quad (\text{A.4})$$

After the transformations $t \rightarrow t$, $s \rightarrow x = s + (b+2)t$, then (A.4) becomes

$$m_t + [(v^2 - v_x^2)m]_x + 2v_x = 0, \quad m = v - v_{xx}. \quad (\text{A.5})$$

Furthermore, by the scaling transformations $v \rightarrow \epsilon^{-1}u$, $t \rightarrow \epsilon^2\tau$, it then follows from (A.5) that

$$m_\tau + [(u^2 - u_x^2)m]_x + 2\epsilon^2 u_x = 0, \quad m = u - u_{xx}. \quad (\text{A.6})$$

Assume that u_x is uniformly bounded in \mathbb{R} and let $\epsilon \rightarrow 0$. Consequently, we arrive at (A.1).

Next, we consider the non-stretching curve flows on the two-dimensional sphere $S^3(R)$, governed by

$$\gamma_t = f\hat{n} + g\hat{t}, \quad (\text{A.6})$$

where \hat{n} and \hat{t} are respectively the normal and tangent vectors, f and g stand for the normal and tangent velocities, depending on the geodesic curvature ϕ of the curve and its derivative with respect to the arc-length s , they satisfy the Frenet equations [3]

$$\begin{pmatrix} \hat{r} \\ \hat{t} \\ \hat{n} \end{pmatrix}_s = \begin{pmatrix} 0 & \rho & 0 \\ -\rho & 0 & \kappa \\ 0 & -\kappa & 0 \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{t} \\ \hat{n} \end{pmatrix}, \quad (\text{A.7})$$

where $\rho = 1/R$ and $\hat{r} = \rho\gamma$ is the unit vector in the radial direction.

Since the Frenet frame is orthonormal, its time evolution is given by

$$\begin{pmatrix} \hat{r} \\ \hat{t} \\ \hat{n} \end{pmatrix}_t = \begin{pmatrix} 0 & \rho V & \rho U \\ -\rho V & 0 & A \\ -\rho U & -A & 0 \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{t} \\ \hat{n} \end{pmatrix}. \quad (\text{A.8})$$

Assume that the curve does not stretch during the curve motion, the arc-length does not depend on time. So s and t can serve as local coordinates on the sphere, and the commute relation

$$\left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right] = 0 \quad (\text{A.9})$$

holds. It follows from (A.7), (A.8) and (A.9) that

$$A = U_s + \phi V$$

and the curvature ϕ satisfies the equation

$$\phi_t = (U_s + \phi V)_s + \rho^2 U \quad (\text{A.10})$$

with the condition

$$V_s = \phi U. \quad (\text{A.11})$$

The substitution of (A.11) into (A.10) leads to the equation for the curvature

$$\phi_t = (\Omega + \rho^2)U, \quad (\text{A.12})$$

where Ω is the recursion operator of the mKdV equation. Set $\phi = m \equiv u - u_{ss}$, $U = -2u_s$, then $V = -(u^2 - u_s^2) + b$, and $u(t, s)$ satisfies the equation

$$m_t + [(u^2 - u_s^2)m]_s + (b+2)u_{sss} - (b-2\rho^2)u_s = 0. \quad (\text{A.13})$$

After the transformations $t \rightarrow t$, $s \rightarrow y = s + (b+2)t$, this equation reduces to

$$m_t + [(u^2 - u_y^2)m]_y + 2(1 + \rho^2)u_y = 0, \quad m = u - u_{yy}. \quad (\text{A.14})$$

Hence following the approximate argument again in (A.5) and (A.6), we obtain (A.1) in a different way.

Remark A.1. *It was shown by Reyes [29] that the CH and HS equations describe pseudo-spherical surfaces. Similarly, we can show that the equation (1.1) also describes pseudo-spherical surfaces, i.e., there exist one-forms*

$$\begin{aligned}
\omega_1 &= \left[\sqrt{\frac{1-\lambda}{1+\lambda}} - \frac{1}{2}(1+\lambda)\sqrt{1-\lambda^2} + \left(\frac{\lambda}{1+\lambda} - \frac{1}{4}\lambda(1+\lambda) \right) m \right] dx \\
&\quad - [2\lambda^{-2}\sqrt{\frac{1-\lambda}{1+\lambda}} + \lambda^{-2}(1+\lambda)\sqrt{1-\lambda^2} + \left(\frac{2}{1+\lambda} + \frac{1+\lambda}{2\lambda} \right) (u_x + \lambda^{-1}u) \\
&\quad + \left(\frac{1}{4}(1+\lambda)\sqrt{1-\lambda^2} + \sqrt{\frac{1-\lambda}{1+\lambda}} + \left(\frac{1}{4}\lambda^{-2}(1+\lambda) + \frac{\lambda}{1+\lambda} \right) m \right) (u^2 - u_x^2)], \\
\omega_2 &= \lambda dx - [2\lambda^{-1} - 2\lambda^{-1}\sqrt{1-\lambda^2}u_x + \lambda(u^2 - u_x^2)]dt, \\
\omega_3 &= \left[-\sqrt{\frac{1-\lambda}{1+\lambda}} - \frac{1}{2}(1+\lambda)\sqrt{1-\lambda^2} - \left(\frac{\lambda}{1+\lambda} + \frac{1}{4}\lambda(1+\lambda) \right) m \right] dx \\
&\quad + [2\lambda^{-2}\sqrt{\frac{1-\lambda}{1+\lambda}} - \lambda^{-2}(1+\lambda)\sqrt{1-\lambda^2} + \left(\frac{2}{1+\lambda} - \frac{1+\lambda}{2\lambda} \right) (u_x + \lambda^{-1}u) \\
&\quad - \left(\frac{1}{4}(1+\lambda)\sqrt{1-\lambda^2} - \sqrt{\frac{1-\lambda}{1+\lambda}} + \left(\frac{1}{4}\lambda^{-2}(1+\lambda) - \frac{\lambda}{1+\lambda} \right) m \right) (u^2 - u_x^2)],
\end{aligned}$$

which satisfy the structure equations for pseudo-spherical surface

$$d\omega_1 = \omega_3 \wedge \omega_2, \quad d\omega_2 = \omega_3 \wedge \omega_2, \quad d\omega_3 = \omega_3 \wedge \omega_2.$$

Based on the structure equations, using the equations for pseudo-potential, we are able to obtain two sets of conservation laws of equation (1.1).

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YING FU

DEPARTMENT OF MATHEMATICS, NORTHWEST UNIVERSITY, XI'AN, 710069, P. R. CHINA

E-mail address: fuying@nwu.edu.cn

GUILONG GUI

DEPARTMENT OF MATHEMATICS, JIANGSU UNIVERSITY, ZHENJIANG, JIANGSU, 212013, P. R. CHINA, AND THE INSTITUTE OF MATHEMATICAL SCIENCES, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, HONG KONG

E-mail address: glgui@amss.ac.cn

YUE LIU

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, ARLINGTON, TX 76019-0408

E-mail address: yliu@uta.edu

CHANGZHENG QU

DEPARTMENT OF MATHEMATICS, NORTHWEST UNIVERSITY, XI'AN, 710069, P. R. CHINA, AND SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455

E-mail address: czqu@nwu.edu.cn

ON THE CAUCHY PROBLEM FOR THE INTEGRABLE CAMASSA-HOLM TYPE EQUATION WITH CUBIC NONLINEARITY

YING FU, GUILONG GUI, YUE LIU, AND CHANGZHENG QU

ABSTRACT. In this paper, we are concerned with the Cauchy problem for the integrable Camassa-Holm type equation with cubic nonlinearity. We establish the local well-posedness in a range of the Besov spaces and derive the blow-up scenario. With analytic initial data, we then show that its solutions are analytic in both variables, globally in space and locally in time. We also demonstrate nonexistence of the smooth traveling wave solutions. Finally, we give geometric descriptions to this integrable equation.

Key words and phrases. Besov space, Local well-posedness, Blow up, Analyticity.
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1. INTRODUCTION

In this paper, we are concerned with the following Cauchy problem of the Camassa-Holm type equation with cubic nonlinearity,

$$\begin{cases} m_t + (u^2 - u_x^2)m_x + 2u_x m^2 = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), \quad m = u - u_{xx}, & x \in \mathbb{R}. \end{cases} \quad (1.1)$$

The equation in (1.1) was introduced by Olver and Rosenau [30] (see also [17]) as a new generalization of integrable system by implementing a simple explicit algorithm based on the bi-Hamiltonian representation of the classically integrable system. In most cases, these new nonlinear systems are endowed with nonlinear dispersion, and thus support non-smooth soliton-like structures. Later, it was obtained again by Qiao [33] from the two-dimensional Euler equation. It was shown in [33] that the equation in (1.1) admits the Lax-pair and the Cauchy problem (1.1) may be solved by the inverse scattering transform method.

The equation in (1.1) is completely integrable and can be rewritten as the bi-Hamiltonian form [30], that is

$$m_t = -((u^2 - u_x^2)m)_x = J \frac{\delta H_0}{\delta m} = K \frac{\delta H_1}{\delta m},$$

where

$$J = -\partial m \partial^{-1} m \partial, \quad \text{and} \quad K = \partial^3 - \partial,$$

corresponding to the Hamiltonian

$$H_0 = 2 \int_{\mathbb{R}} m u \, dx,$$

and the Hamiltonian

$$H_1 = \frac{1}{4} \int_{\mathbb{R}} \left(u^4 + 2u^2 u_x^2 - \frac{1}{3} u_x^4 \right) dx.$$

It also admits the Lax pair [33], that is

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = U(m, \lambda) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = V(m, u, \lambda) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

where

$$U(m, \lambda) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \lambda m \\ -\frac{1}{2} \lambda m & \frac{1}{2} \end{pmatrix}, \quad m = u - u_{xx},$$

and

$$V(m, u, \lambda) = \begin{pmatrix} \lambda^{-2} + \frac{1}{2}(u^2 - u_x^2) & -\lambda^{-1}(u - u_x) - \frac{1}{2} \lambda m(u^2 - u_x^2) \\ \lambda^{-1}(u + u_x) + \frac{1}{2} \lambda m(u^2 - u_x^2) & -\lambda^{-2} - \frac{1}{2}(u^2 - u_x^2) \end{pmatrix}.$$

The Camassa-Holm (CH) equation [2, 18] defined by

$$m_t + um_x + 2u_x m = 0, \quad m = u - u_{xx}$$

has attracted much attention in the last twenty years because of its interesting properties: complete integrability, existence of peaked solitons and multi-peakons [2, 3], geometric formulations [6, 12, 25, 27] and the presence of breaking waves (i.e. a solution that remains bounded while its slope becomes unbounded in finite time) [8, 9, 10, 11]. Note that the nonlinearity in the CH equation is quadratic. In contrast to the integrable modified KdV equation with a cubic nonlinearity, it is our interest to find an integrable CH-type equations with a cubic nonlinearity. Indeed, two integrable CH-type equations with cubic nonlinearity have been discovered recently. One is the equation (1.1) and the second one is the so-called Novikov equation [29]. The integrability, peaked solitons, well-posedness and blow up phenomena to the Novikov equation have been studied extensively, see the ref. [22, 23, 29, 36, 37].

The goal of the present paper is to establish qualitative results for the initial value problem (1.1). We first study the local well-posedness for the strong solutions to the Cauchy problem (1.1). The proof of the local well-posedness is inspired by the argument of approximate solutions by Danchin [15] in the study of the local well-posedness to the CH equation. However, one problematic issue is that we here deal with a higher order nonlinearity in the Besov spaces, making the proof of several required nonlinear estimates somewhat delicate. These difficulties are nevertheless overcome by carefully estimates for each iterative approximation of solutions to (1.1). With the local well-posedness obtained in hand, we then present a precise blow-up scenario and a conservative property. We also prove the analyticity of its solutions $u = u(t, x)$ in both variables, with x in \mathbb{R} and t in an interval around zero, provided that the initial profile u_0 is an analytic function on the real line. Hence, this analytic result can be viewed as a Cauchy-Kowalevski theorem for (1.1). Finally, we give the geometric descriptions to this equation.

It is well known that the solutions of the KdV equation are analytic in the space variable for all time [38] but are not analytic in the time variable [24]. In contrast with the KdV equation, the solutions to the Hunter-Saxton (HS) and Camassa-Holm equations are analytic in both space and time variables for a short time

[21, 36]. Like the CH and HS equations, we will show that solutions of the Cauchy problem (1.1) are analytic in both space and time variables.

As mentioned above, a well known fact is that the CH equation has the peakons [2], which were shown to be orbitally stable in the intriguing papers [13, 14]. Stability of the periodic peakons of the CH equation can be found in [26]. So it is of interest to identify traveling-wave solutions of the equation in (1.1). Indeed, Qiao [33] found that the equation in (1.1) has a cusped soliton given by

$$u(t, x) = u(X) = \pm \left(2 - 3 \cosh^2 X + \left(\cosh X + \frac{1}{3} \right) \sqrt{3(3 \cosh X + 1)(\cosh X - 1)} \right),$$

where $X = \frac{x}{2} - \frac{11}{6}t$, and a so called "W/M" -shape-peakon soliton of the form

$$u(t, x) = u(X) = 2 - 3 \cosh^2 X + \left(\cosh X + \frac{1}{3} \right) \sqrt{3(3 \cosh X + 1)(\cosh X - 1)},$$

where $X = \frac{|x - \frac{11}{3}t|}{2} - \ln 2$. In this paper, we are able to show that the equation in (1.1) does not have nontrivial smooth traveling-wave solutions.

The plan of the paper is as follows. In Section 2, some preliminary properties, which will be used later, are presented. The local well-posedness in the Besov spaces is established in Section 3. In Section 4, a blow-up scenario and a global conservative property of (1.1) will be derived. Section 5 is devoted to the study of the analyticity of the Cauchy problem (1.1) based on a contraction type argument in a suitably chosen scale of the Banach spaces. Such an approach to analytic regularity of solutions to Cauchy problem (1.1) was initiated by Ovsjannikov [31, 32] as an abstract Cauchy-Kowalevski theorem and later further developed by Nirenberg [28], Baouendi and Goulaouic [1] among others and subsequently applied to the Euler and Navier-Stokes equation. Non-existence of the smooth traveling wave solutions is proved in Section 6. In Appendix A, the precise geometric descriptions of the equation in (1.1) are given.

Notation. In the following, for a given Banach space Z , we denote its norm by $\|\cdot\|_Z$. Since all space of functions are over \mathbb{R} , for simplicity, we drop \mathbb{R} in our notations of function spaces if there is no ambiguity. We denote $\mathcal{F}u$ or \hat{u} the Fourier transform of the function u .

2. PRELIMINARIES

In this section, we first present the Littlewood-Paley theory and the properties of the Besov-Sobolev spaces which will play a key role to prove local well-posedness for the Cauchy problem (1.1). Then we introduce a new space, whose properties are studied. In order to verify the analyticity of solutions to (1.1), the abstract Cauchy-Kowalevski theorem for identifying analyticity of the Cauchy problem is presented.

Proposition 2.1. [5] (*Littlewood-Paley decomposition*) Let $\mathcal{B} \triangleq \{\xi \in \mathbb{R}, |\xi| \leq \frac{4}{3}\}$ and $\mathcal{C} \triangleq \{\xi \in \mathbb{R}, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. There exist two radial functions $\chi \in C_c^\infty(\mathcal{B})$ and $\varphi \in C_c^\infty(\mathcal{C})$ such that

$$\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1, \quad \forall \quad \xi \in \mathbb{R}^d,$$

$$\begin{aligned} |q - q'| \geq 2 &\Rightarrow \text{Supp}\varphi(2^{-q}\cdot) \cap \text{Supp}\varphi(2^{-q'}\cdot) = \emptyset, \\ q \geq 1 &\Rightarrow \text{Supp}\chi(\cdot) \cap \text{Supp}\varphi(2^{-q}\cdot) = \emptyset, \end{aligned}$$

and

$$\frac{1}{3} \leq \chi(\xi)^2 + \sum_{q \geq 0} \varphi((2^{-q}\xi))^2 \leq 1, \quad \forall \quad \xi \in \mathbb{R}^d.$$

Furthermore, let $h \triangleq \mathcal{F}^{-1}\varphi$ and $\tilde{h} \triangleq \mathcal{F}^{-1}\chi$. Then the dyadic operators Δ_q and S_q can be defined as follows

$$\begin{aligned} \Delta_q f &\triangleq \varphi(2^{-q}D)f = 2^{qd} \int_{\mathbb{R}^d} h(2^q y) f(x-y) dy \quad \text{for } q \geq 0, \\ S_q f &\triangleq \chi(2^{-q}D)f = \sum_{-1 \leq k \leq q-1} \Delta_k f = 2^{qd} \int_{\mathbb{R}^d} \tilde{h}(2^q y) f(x-y) dy, \\ \Delta_{-1} f &\triangleq S_0 f \text{ and } \Delta_q f \triangleq 0 \quad \text{for } q \leq -2. \end{aligned}$$

Lemma 2.1. [5] (*Bernstein's inequality*) Let \mathcal{B} be a ball with center 0 in \mathbb{R}^d and \mathcal{C} a ring with center 0 in \mathbb{R}^d . A constant C exists so that, for any positive real number λ , any non negative integer k , any smooth homogeneous function σ of degree m and any couple of real numbers (a, b) with $b \geq a \geq 1$, there hold

$$\begin{aligned} \text{Supp} \hat{u} \subset \lambda \mathcal{B} &\Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^{k+1} \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a}; \\ \text{Supp} \hat{u} \subset \lambda \mathcal{C} &\Rightarrow C^{-1-k} \lambda^k \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^{1+k} \lambda^k \|u\|_{L^a}; \\ \text{Supp} \hat{u} \subset \lambda \mathcal{C} &\Rightarrow \|\sigma(D)u\|_{L^b} \leq C_{\sigma,m} \lambda^{m+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a}. \end{aligned}$$

for any function $u \in L^a$.

Definition 2.1. [5] (*Besov space*) Let $s \in \mathbb{R}, 1 \leq p, r \leq \infty$. The inhomogenous Besov space $B_{p,r}^s(\mathbb{R}^d)$ ($B_{p,r}^s$ for short) is defined by

$$B_{p,r}^s \triangleq \{f \in \mathcal{S}'(\mathbb{R}^d); \quad \|f\|_{B_{p,r}^s} < \infty\},$$

where

$$\|f\|_{B_{p,r}^s} \triangleq \begin{cases} \left(\sum_{q \in \mathbb{Z}} 2^{qs} \|\Delta_q f\|_{L^p}^r \right)^{\frac{1}{r}}, & \text{for } r < \infty, \\ \sup_{q \in \mathbb{Z}} 2^{qs} \|\Delta_q f\|_{L^p}, & \text{for } r = \infty. \end{cases}$$

If $s = \infty$, $B_{p,r}^\infty \triangleq \bigcap_{s \in \mathbb{R}} B_{p,r}^s$.

Proposition 2.2. [15, 16] *The following properties hold.*

- i) *Density: if $p, r < \infty$, then $\mathcal{S}(\mathbb{R}^d)$ is dense in $B_{p,r}^s(\mathbb{R}^d)$.*
- ii) *Sobolev embeddings: if $p_1 \leq p_2$ and $r_1 \leq r_2$, then $B_{p_1,r_1}^s \hookrightarrow B_{p_2,r_2}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}$. If $s_1 < s_2$, $1 \leq p \leq +\infty$ and $1 \leq r_1, r_2 \leq +\infty$, then the embedding $B_{p,r_2}^{s_2} \hookrightarrow B_{p,r_1}^{s_1}$ is locally compact.*
- iii) *Algebraic properties: for $s > 0$, $B_{p,r}^s \cap L^\infty$ is an algebra. Moreover, $(B_{p,r}^s \text{ is an algebra}) \iff (B_{p,r}^s \hookrightarrow L^\infty) \iff (s > \frac{d}{p} \text{ or } (s \geq \frac{d}{p} \text{ and } r = 1))$.*
- iv) *Fatou property: if $(u^{(n)})_{n \in \mathbb{N}}$ is a bounded sequence of $B_{p,r}^s$ which tends to u in \mathcal{S}' , then $u \in B_{p,r}^s$ and*

$$\|u\|_{B_{p,r}^s} \leq \liminf_{n \rightarrow \infty} \|u^{(n)}\|_{B_{p,r}^s}.$$

v) *Complex interpolation:* if $u \in B_{p,r}^s \cap B_{p,r}^{\bar{s}}$ and $\theta \in [0, 1]$, $1 \leq p, r \leq \infty$, then $u \in B_{p,r}^{\theta s + (1-\theta)\bar{s}}$ and $\|u\|_{B_{p,r}^{\theta s + (1-\theta)\bar{s}}} \leq \|u\|_{B_{p,r}^s}^\theta \|u\|_{B_{p,r}^{\bar{s}}}^{1-\theta}$.

vi) Let $m \in \mathbb{R}$ and f be a S^m -multiplier (that is, $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is smooth and satisfies that for all multi-index α , there exists a constant C_α such that for any $\xi \in \mathbb{R}^d$, $|\partial^\alpha f(\xi)| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}$.) Then for all $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, the operator $f(D)$ is continuous from $B_{p,r}^s$ to $B_{p,r}^{s-m}$.

Lemma 2.2. [15, 16] Suppose that $(p, r) \in [1, +\infty]^2$ and $s > -\frac{d}{p}$. Let v be a vector field such that ∇v belongs to $L^1([0, T]; B_{p,r}^{s-1})$ if $s > 1 + \frac{d}{p}$ or to $L^1([0, T]; B_{p,r}^{\frac{d}{p}} \cap L^\infty)$ otherwise. Suppose also that $f_0 \in B_{p,r}^s$, $F \in L^1([0, T]; B_{p,r}^s)$ and that $f \in L^\infty([0, T]; B_{p,r}^s) \cap C([0, T]; \mathcal{S}')$ solves the d -dimensional linear transport equations

$$(T) \quad \begin{cases} \partial_t f + v \cdot \nabla f = F, \\ f|_{t=0} = f_0. \end{cases}$$

Then there exists a constant C depending only on s, p and d such that the following statements hold:

1) If $r = 1$ or $s \neq 1 + \frac{d}{p}$, then

$$\|f\|_{B_{p,r}^s} \leq \|f_0\|_{B_{p,r}^s} + \int_0^t \|F(\tau)\|_{B_{p,r}^s} d\tau + C \int_0^t V'(\tau) \|f(\tau)\|_{B_{p,r}^s} d\tau,$$

or

$$\|f\|_{B_{p,r}^s} \leq e^{CV(t)} \left(\|f_0\|_{B_{p,r}^s} + \int_0^t e^{-CV(\tau)} \|F(\tau)\|_{B_{p,r}^s} d\tau \right) \quad (2.1)$$

hold, where $V(t) = \int_0^t \|\nabla v(\tau)\|_{B_{p,r}^{\frac{d}{p}} \cap L^\infty} d\tau$ if $s < 1 + \frac{d}{p}$ and $V(t) = \int_0^t \|\nabla v(\tau)\|_{B_{p,r}^{s-1}} d\tau$ else.

2) If $s \leq 1 + \frac{d}{p}$ and, in addition, $\nabla f_0 \in L^\infty$, $\nabla f \in L^\infty([0, T] \times \mathbb{R}^d)$ and $\nabla F \in L^1([0, T]; L^\infty)$, then

$$\begin{aligned} & \|f(t)\|_{B_{p,r}^s} + \|\nabla f(t)\|_{L^\infty} \\ & \leq e^{CV(t)} \left(\|f_0\|_{B_{p,r}^s} + \|\nabla f_0\|_{L^\infty} + \int_0^t e^{-CV(\tau)} (\|F(\tau)\|_{B_{p,r}^s} + \|\nabla F(\tau)\|_{L^\infty}) d\tau \right) \end{aligned}$$

with $V(t) = \int_0^t \|\nabla v(\tau)\|_{B_{p,r}^{\frac{d}{p}} \cap L^\infty} d\tau$.

3) If $f = v$, then for all $s > 0$, the estimate (2.1) holds with $V(t) = \int_0^t \|\partial_x u(\tau)\|_{L^\infty} d\tau$.

4) If $r < +\infty$, then $f \in C([0, T]; B_{p,r}^s)$. If $r = +\infty$, then $f \in C([0, T]; B_{p,1}^{s'})$ for all $s' < s$.

Lemma 2.3. [16] Let $(p, p_1, r) \in [1, +\infty]^3$. Assume that $s > -d \min\{\frac{1}{p_1}, \frac{1}{p'}\}$ with $p' \triangleq (1 - \frac{1}{p})^{-1}$. Let $f_0 \in B_{p,r}^s$ and $F \in L^1([0, T]; B_{p,r}^s)$. Let v be a time dependent vector field such that $v \in L^\rho([0, T]; B_{\infty,\infty}^{-M})$ for some $\rho > 1$, $M > 0$ and $\nabla v \in L^1([0, T]; B_{p_1,\infty}^{\frac{d}{p_1}} \cap L^\infty)$ if $s < 1 + \frac{d}{p_1}$, and $\nabla v \in L^1([0, T]; B_{p_1,r}^{s-1})$ if $s > 1 + \frac{d}{p_1}$ or $s = 1 + \frac{d}{p_1}$ and $r = 1$. Then the transport equations (T) has a unique solution $f \in L^\infty([0, T]; B_{p,r}^s) \cap (\cap_{s' < s} C([0, T]; B_{p,1}^{s'}))$ and the inequalities in Lemma 2.2 hold true. If, moreover, $r < \infty$, then we have $f \in C([0, T]; B_{p,r}^s)$.

Lemma 2.4. [5] (*1-D Moser-type estimates*) Assume that $1 \leq p, r \leq +\infty$, the following estimates hold:

- (i) for $s > 0$, $\|fg\|_{B_{p,r}^s(\mathbb{R})} \leq C(\|f\|_{B_{p,r}^s(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})} + \|g\|_{B_{p,r}^s(\mathbb{R})}\|f\|_{L^\infty(\mathbb{R})});$
- (ii) for $s_1 \leq \frac{1}{p}$, $s_2 > \frac{1}{p}$ ($s_2 \geq \frac{1}{p}$ if $r = 1$) and $s_1 + s_2 > 0$,

$$\|fg\|_{B_{p,r}^{s_1}(\mathbb{R})} \leq C\|f\|_{B_{p,r}^{s_1}(\mathbb{R})}\|g\|_{B_{p,r}^{s_2}(\mathbb{R})},$$

where C 's are constants independent of f and g .

In order to apply the contraction argument to the proof of the analytic regularity of the Cauchy problem (1.1), we need a suitable scale of Banach spaces which we proceed to describe below.

For any $s > 0$, we set

$$E_s = \left\{ u \in C^\infty(\mathbb{R}) : |||u|||_s = \sup_{k \in N_0} \frac{s^k \|\partial^k u\|_{H^2}}{k!/(k+1)^2} < \infty \right\},$$

where $H^2(\mathbb{R})$ is the Sobolev space of order two on the real line and N_0 is the set of nonnegative integers. One can easily verify that E_s equipped with the norm $|||\cdot|||_s$ is a Banach space and that, for any $0 < s' < s$, E_s is continuously embedded in $E_{s'}$ with

$$|||u|||_{s'} \leq |||u|||_s.$$

Another simple consequence of the definition is that any u in E_s is a real analytic function on \mathbb{R} . Crucial for our purposes is the fact that each E_s forms an algebra under pointwise multiplication of functions.

Lemma 2.5. [21] Let $0 < s < 1$. There is a constant $c > 0$, independent of s , such that for any u and v in E_s we have

$$|||uv|||_s \leq c|||u|||_s|||v|||_s.$$

Lemma 2.6. [21] There is a constant $c > 0$ such that for any $0 < s' < s < 1$, we have

$$|||\partial_x u|||_{s'} \leq \frac{c}{s-s'}|||u|||_s,$$

and

$$|||(1 - \partial_x^2)^{-1}u|||_{s'} \leq |||u|||_s, \quad |||(1 - \partial_x^2)^{-1}\partial_x u|||_{s'} \leq |||u|||_s.$$

The following theorem comes from [1].

Theorem 2.1. [1] Let $\{X_s\}_{0 < s < 1}$ be a scale of decreasing Banach spaces, namely for any $s' < s$ we have $X_s \subset X_{s'}$ and $|||\cdot|||_{s'} \leq |||\cdot|||_s$. Consider the Cauchy problem

$$\begin{cases} \frac{du}{dt} = F(t, u(t)), \\ u(0) = 0. \end{cases} \quad (2.2)$$

Let T, R and C be positive constants and assume that F satisfies the following conditions

- 1) If for $0 < s' < s < 1$ the function $t \mapsto u(t)$ is holomorphic in $|t| < T$ and continuous on $|t| \leq T$ with values in X_s and

$$\sup_{|t| \leq T} |||u(t)|||_s < R,$$

then $t \mapsto F(t, u(t))$ is a holomorphic function on $|t| < T$ with values in $X_{s'}$.

2) For any $0 < s' < s < 1$ and any $u, v \in X_s$ with $\|u\|_s < R$, $\|v\|_s < R$,

$$\sup_{|t| \leq T} \|F(t, u) - F(t, v)\|_{s'} \leq \frac{C}{s - s'} \|u - v\|_s.$$

3) There exists $M > 0$ such that for any $0 < s < 1$,

$$\sup_{|t| \leq T} \|F(t, 0)\|_s \leq \frac{M}{1 - s}.$$

Then there exists a $T_0 \in (0, T)$ and a unique function $u(t)$, which for every $s \in (0, 1)$ is holomorphic in $|t| < (1 - s)T_0$ with values in X_s , and is a solution to the Cauchy problem (2.2).

Next we restate the Cauchy problem (1.1) in a more convenient form. Note that (1.1) is equivalent to the following one.

$$u_t - u_{xxt} + 3u^2u_x - 4uu_xu_{xx} + u_x^2u_{xxx} + 2u_xu_{xx}^2 - u^2u_{xxx} - u_x^3 = 0.$$

Applying the operator $(1 - \partial_x^2)^{-1}$ to both sides of the above equation, we obtain

$$u_t + \left(u^2 - \frac{1}{3}u_x^2\right)u_x + \partial_x(1 - \partial_x^2)^{-1} \left(\frac{2}{3}u^3 + uu_x^2\right) + (1 - \partial_x^2)^{-1} \frac{u_x^3}{3} = 0.$$

Differentiating with respect to x on both sides of the above equation and noticing that $\partial_x^2(1 - \partial_x^2)^{-1} = (1 - \partial_x^2)^{-1} - I$, one finds that

$$u_{tx} + u^2u_{xx} + uu_x^2 - u_x^2u_{xx} - \frac{2}{3}u^3 + (1 - \partial_x^2)^{-1} \left(\frac{2}{3}u^3 + uu_x^2\right) + \partial_x(1 - \partial_x^2)^{-1} \frac{u_x^3}{3} = 0.$$

Let $v = u_x$. Then the problem (1.1) can be written as a system for u and v :

$$\begin{cases} u_t = -u^2v + \frac{1}{3}v^3 - \partial_x(1 - \partial_x^2)^{-1} \left(\frac{2}{3}u^3 + uv^2\right) - (1 - \partial_x^2)^{-1} \frac{v^3}{3}, \\ v_t = -uv^2 + \frac{2}{3}u^3 + v^2v_x - u^2v_x - (1 - \partial_x^2)^{-1} \left(\frac{2}{3}u^3 + uv^2\right) - \partial_x(1 - \partial_x^2)^{-1} \frac{v^3}{3}, \\ u(0, x) = u_0(x), \quad v(0, x) = u'_0(x). \end{cases}$$

Note that the initial data in the Cauchy problem (2.2) of the abstract Cauchy-Kowalevski theorem equals to zero, one can set $U = u - u_0$, $V = v - u'_0$, then the above problem is equivalent to

$$\begin{cases} U_t = -(U + u_0)^2(V + u'_0) - \partial_x(1 - \partial_x^2)^{-1} \left(\frac{2}{3}(U + u_0)^3 + (U + u_0)(V + u'_0)^2\right) \\ \quad - (1 - \partial_x^2)^{-1} \frac{1}{3}(V + u'_0)^3 + \frac{1}{3}(V + u'_0)^3 = F(U, V), \\ V_t = -(U + u_0)(V + u'_0)^2 + \frac{2}{3}(U + u_0)^3 + (V + u'_0)^2(V + u'_0)_x \\ \quad - (1 - \partial_x^2)^{-1} \left(\frac{2}{3}(U + u_0)^3 + (U + u_0)(V + u'_0)^2\right) \\ \quad - \partial_x(1 - \partial_x^2)^{-1} \frac{1}{3}(V + u'_0)^3 - (U + u_0)^2(V + u'_0)_x = G(U, V), \\ U(0, x) = 0, \quad V(0, x) = 0. \end{cases} \quad (2.3)$$

This can be written as the abstract form of the Cauchy problem in (2.2).

3. LOCAL WELL-POSEDNESS

In this section, we shall discuss the local well-posedness of the Cauchy problem (1.1). At first, we present the following definition.

Definition 3.1. For $T > 0$, $s \in \mathbb{R}$ and $1 \leq p \leq +\infty$, we set

$$\begin{aligned} E_{p,r}^s(T) &\triangleq \mathcal{C}([0, T]; B_{p,r}^s) \cap \mathcal{C}^1([0, T]; B_{p,r}^{s-1}) \quad \text{if } r < +\infty, \\ E_{p,\infty}^s(T) &\triangleq L^\infty([0, T]; B_{p,\infty}^s) \cap Lip([0, T]; B_{p,\infty}^{s-1}) \end{aligned}$$

and $E_{p,r}^s \triangleq \cap_{T>0} E_{p,r}^s(T)$.

Our main local existence result is the following theorem.

Theorem 3.1. Suppose that $1 \leq p, r \leq +\infty$, $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$ and $u_0 \in B_{p,r}^s$. Then there exists a time $T > 0$ such that the initial-value problem (1.1) has a unique solution $u \in E_{p,r}^s(T)$, and the map $u_0 \mapsto u$ is continuous from a neighborhood of u_0 in $B_{p,r}^s$ into

$$\mathcal{C}([0, T]; B_{p,r}^{s'}) \cap \mathcal{C}^1([0, T]; B_{p,r}^{s'-1})$$

for every $s' < s$ when $r = +\infty$ and $s' = s$ whereas $r < +\infty$.

Remark 3.1. When $p = r = 2$, the Besov space $B_{p,r}^s(\mathbb{R})$ coincides with the Sobolev space $H^s(\mathbb{R})$. Theorem 3.1 implies that under the condition $u_0 \in H^s(\mathbb{R})$ with $s > 5/2$, we can obtain the local well-posedness for the initial-value problem (1.1).

Remark 3.2. The existence time for the initial-value problem (1.1) may be chosen independently of s in the following sense [39]. If

$$u \in \mathcal{C}([0, T]; H^s) \cap \mathcal{C}^1([0, T]; H^{s-1})$$

is the solution of the initial-value problem (1.1) with initial data $u_0 \in H^r$ for some $r > 5/2$, $r \neq s$, then

$$u \in \mathcal{C}([0, T]; H^r) \cap \mathcal{C}^1([0, T]; H^{r-1})$$

with the same time T . In particular, if $u_0 \in H^\infty$, then $u \in C([0, T]; H^\infty)$.

In the following, we denote $C > 0$ a generic constant only depending on p, r, s . Uniqueness and continuity with respect to the initial data are an immediate consequence of the following result.

Proposition 3.1. Let $1 \leq p, r \leq +\infty$ and $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$. Let $u^{(1)}, u^{(2)}$ be two given solutions of the initial-value problem (1.1) with the initial data $u_0^{(1)}, u_0^{(2)} \in B_{p,r}^s$ satisfying $u^{(1)}, u^{(2)} \in L^\infty([0, T]; B_{p,r}^s) \cap \mathcal{C}([0, T]; \mathcal{S}')$. Then for every $t \in [0, T]$:

$$\begin{aligned} &\|u^{(1)}(t) - u^{(2)}(t)\|_{B_{p,r}^{s-1}} \\ &\leq \|u_0^{(1)} - u_0^{(2)}\|_{B_{p,r}^{s-1}} \exp \left\{ C \int_0^t (\|u^{(1)}(\tau)\|_{B_{p,r}^s}^2 + \|u^{(2)}(\tau)\|_{B_{p,r}^s}^2) d\tau \right\}. \end{aligned} \quad (3.1)$$

Proof. Denote $u^{(12)} \triangleq u^{(2)} - u^{(1)}$ and $m^{(12)} \triangleq m^{(2)} - m^{(1)}$. It is obvious that

$$u^{(12)} \in L^\infty([0, T]; B_{p,r}^s) \cap \mathcal{C}([0, T]; \mathcal{S}'),$$

which implies that $u^{(12)} \in \mathcal{C}([0, T]; B_{p,r}^{s-1})$ and $u^{(12)}, m^{(12)}$ solves the transport equation

$$\begin{cases} \partial_t m^{(12)} + [(u^{(1)})^2 - (u_x^{(1)})^2] \partial_x m^{(12)} = f(u^{(12)}, m^{(12)}, u^{(1)}, u^{(2)}, m^{(1)}, m^{(2)}), \\ m^{(12)}|_{t=0} = m_0^{(12)} \triangleq m_0^{(2)} - m_0^{(1)}, \end{cases}$$

with

$$f(u^{(12)}, m^{(12)}, u^{(1)}, u^{(2)}, m^{(1)}, m^{(2)}) := -[(u^{(12)}(u^{(1)} + u^{(2)}) - u_x^{(12)}(u_x^{(1)} + u_x^{(2)}))m_x^{(2)} - 2u_x^{(1)}m^{(12)}(m^{(1)} + m^{(2)}) - 2u_x^{(12)}(m^{(2)})^2].$$

According to Lemma 2.2, we have

$$\begin{aligned} & e^{-C \int_0^t \|\partial_x [(u^{(1)})^2 - (u_x^{(1)})^2](\tau)\|_{B_{p,r}^{s-2}} d\tau} \|m^{(12)}(t)\|_{B_{p,r}^{s-3}} \\ & \leq \|m_0^{(12)}\|_{B_{p,r}^{s-3}} + C \int_0^t e^{-C \int_0^\tau \|\partial_x [(u^{(1)})^2 - (u_x^{(1)})^2](\tau')\|_{B_{p,r}^{s-2}} d\tau'} \\ & \quad \times \|f(u^{(12)}, m^{(12)}, u^{(1)}, u^{(2)}, m^{(1)}, m^{(2)})(\tau)\|_{B_{p,r}^{s-3}} d\tau. \end{aligned} \quad (3.2)$$

For $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$, by Lemma 2.4 we have

$$\begin{aligned} & \| [u^{(12)}(u^{(1)} + u^{(2)}) - u_x^{(12)}(u_x^{(1)} + u_x^{(2)})] m_x^{(2)} \|_{B_{p,r}^{s-3}} \\ & \leq C \|m^{(2)}\|_{B_{p,r}^{s-2}} \|u^{(12)}\|_{B_{p,r}^{s-1}} (\|u^{(1)}\|_{B_{p,r}^{s-1}} + \|u^{(2)}\|_{B_{p,r}^{s-1}}) \\ & \leq C \|u^{(12)}\|_{B_{p,r}^{s-1}} (\|u^{(1)}\|_{B_{p,r}^s}^2 + \|u^{(2)}\|_{B_{p,r}^s}^2), \\ & \|u_x^{(1)} m^{(12)}(m^{(1)} + m^{(2)})\|_{B_{p,r}^{s-3}} \\ & \leq C \|u^{(1)}\|_{B_{p,r}^{s-1}} \|m^{(12)}\|_{B_{p,r}^{s-3}} (\|m^{(1)}\|_{B_{p,r}^{s-2}} + \|m^{(2)}\|_{B_{p,r}^{s-2}}) \\ & \leq C \|u^{(12)}\|_{B_{p,r}^{s-1}} (\|u^{(1)}\|_{B_{p,r}^s}^2 + \|u^{(2)}\|_{B_{p,r}^s}^2), \end{aligned}$$

and

$$\|u_x^{(12)}(m^{(2)})^2\|_{B_{p,r}^{s-3}} \leq C \|u^{(12)}\|_{B_{p,r}^{s-2}} \|m^{(2)}\|_{B_{p,r}^{s-2}}^2 \leq C \|u^{(12)}\|_{B_{p,r}^{s-1}} \|u^{(2)}\|_{B_{p,r}^s}^2.$$

Therefore, inserting the above estimates to (3.2) we obtain

$$\begin{aligned} & e^{-C \int_0^t \|\partial_x [(u^{(1)})^2 - (u_x^{(1)})^2](\tau)\|_{B_{p,r}^{s-2}} d\tau} \|u^{(12)}(t)\|_{B_{p,r}^{s-1}} \\ & \leq \|u_0^{(12)}\|_{B_{p,r}^{s-1}} + C \int_0^t e^{-C \int_0^\tau \|\partial_x [(u^{(1)})^2 - (u_x^{(1)})^2](\tau')\|_{B_{p,r}^{s-2}} d\tau'} \\ & \quad \times \|u^{(12)}(\tau)\|_{B_{p,r}^{s-1}} (\|u^{(1)}\|_{B_{p,r}^s}^2 + \|u^{(2)}\|_{B_{p,r}^s}^2) d\tau. \end{aligned}$$

Hence, thanks to

$$\|\partial_x [(u^{(1)})^2 - (u_x^{(1)})^2]\|_{B_{p,r}^{s-2}} \leq C (\|u^{(1)}\|_{B_{p,r}^s}^2 + \|u^{(2)}\|_{B_{p,r}^s}^2)$$

and then applying Gronwall's inequality, we reach (3.1). \square

Now let us start the proof of Theorem 3.1, which is motivated by the proof of local existence theorem about the Camassa-Holm equation in [15]. Firstly, we shall use the classical Friedrichs regularization method to construct the approximate solutions to the Cauchy problem problem (1.1).

Lemma 3.1. *Let u_0, p, r and s be as in the statement of Theorem 3.1. Assume that $u^{(0)} := 0$. There exists a sequence of smooth functions $(u^{(n)})_{n \in \mathbb{N}} \in \mathcal{C}(\mathbb{R}^+; B_{p,r}^\infty)$ solving the following linear transport equation by induction:*

$$(T_n) \quad \begin{cases} \{\partial_t + [(u^{(n)})^2 - (u_x^{(n)})^2] \partial_x\} m^{(n+1)} = -2u_x^{(n)} (m^{(n)})^2, & t > 0, x \in \mathbb{R}, \\ u^{(n+1)}|_{t=0} = u_0^{(n+1)}(x) = S_{n+1} u_0, & x \in \mathbb{R}. \end{cases} \quad (3.3)$$

Moreover, there exists a $T > 0$ such that the solutions satisfying the following properties:

- (i) $(u^{(n)})_{n \in \mathbb{N}}$ is uniformly bounded in $E_{p,r}^s(T)$.
- (ii) $(u^{(n)})_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0, T]; B_{p,r}^{s-1})$.

Proof. Since all data $S_{n+1} u_0$ belongs to $B_{p,r}^\infty$, Lemma 2.3 enables us to show by induction that for all $n \in \mathbb{N}$, the equation (T_n) has a global solution which belongs to $\mathcal{C}(\mathbb{R}; B_{p,r}^\infty)$. Thanks to Lemma 2.2 and the proof of Proposition 3.1, we have the following inequality for all $n \in \mathbb{N}$:

$$\begin{aligned} & e^{-C \int_0^t \|\partial_x [(u^{(n)})^2 - (u_x^{(n)})^2](\tau)\|_{B_{p,r}^{s-2}} d\tau} \|m^{(n+1)}(t)\|_{B_{p,r}^{s-2}} \\ & \leq \|m_0\|_{B_{p,r}^{s-2}} + C \int_0^t e^{-C \int_0^\tau \|\partial_x [(u^{(n)})^2 - (u_x^{(n)})^2](\tau')\|_{B_{p,r}^{s-2}} d\tau'} \|u_x^{(n)} (m^{(n)})^2\|_{B_{p,r}^{s-2}} d\tau. \end{aligned}$$

Thanks to $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$, we find $B_{p,r}^{s-2}$ is an algebra. From this, one obtains

$$\|u_x^{(n)} (m^{(n)})^2\|_{B_{p,r}^{s-2}} \leq C \|u_x^{(n)}\|_{B_{p,r}^{s-2}} \|m^{(n)}\|_{B_{p,r}^{s-2}}^2 \leq C \|u^{(n)}\|_{B_{p,r}^s}^3,$$

which along with the above inequality leads to

$$\begin{aligned} & e^{-C \int_0^t \|\partial_x [(u^{(n)})^2 - (u_x^{(n)})^2](\tau)\|_{B_{p,r}^{s-2}} d\tau} \|u^{(n+1)}(t)\|_{B_{p,r}^s} \\ & \leq \|u_0\|_{B_{p,r}^s} + C \int_0^t e^{-C \int_0^\tau \|\partial_x [(u^{(n)})^2 - (u_x^{(n)})^2](\tau')\|_{B_{p,r}^{s-2}} d\tau'} \|u^{(n)}(\tau)\|_{B_{p,r}^s}^3 d\tau. \end{aligned}$$

Hence, we get

$$\begin{aligned} \|u^{(n+1)}(t)\|_{B_{p,r}^s} & \leq e^{C \int_0^t \|\partial_x [(u^{(n)})^2 - (u_x^{(n)})^2](\tau)\|_{B_{p,r}^{s-2}} d\tau} \|u_0\|_{B_{p,r}^s} \\ & \quad + C \int_0^t e^{C \int_\tau^t \|\partial_x [(u^{(n)})^2 - (u_x^{(n)})^2](\tau')\|_{B_{p,r}^{s-2}} d\tau'} \|u^{(n)}(\tau)\|_{B_{p,r}^s}^3 d\tau. \end{aligned} \quad (3.4)$$

Let us choose a $T > 0$ such that $4C \|u_0\|_{B_{p,r}^s}^2 T < 1$, and suppose by induction that for all $t \in [0, T]$

$$\|u^{(n)}(t)\|_{B_{p,r}^s} \leq \frac{\|u_0\|_{B_{p,r}^s}}{(1 - 4C \|u_0\|_{B_{p,r}^s}^2 t)^{1/2}}. \quad (3.5)$$

Indeed, since $B_{p,r}^{s-2}$ is an algebra, one obtains from (3.5) that for any $0 \leq \tau \leq t$

$$\begin{aligned} & C \int_\tau^t \|\partial_x [(u^{(n)})^2 - (u_x^{(n)})^2](\tau')\|_{B_{p,r}^{s-2}} d\tau' \leq C \int_\tau^t \|u^{(n)}(\tau')\|_{B_{p,r}^s}^2 d\tau' \\ & \leq C \int_\tau^t \frac{\|u_0\|_{B_{p,r}^s}^2}{1 - 4C \|u_0\|_{B_{p,r}^s}^2 \tau'} d\tau' = \frac{1}{4} \ln(1 - 4C \|u_0\|_{B_{p,r}^s}^2 \tau) - \frac{1}{4} \ln(1 - 4C \|u_0\|_{B_{p,r}^s}^2 t). \end{aligned}$$

And then inserting the above inequality and (3.5) into (3.4) leads to

$$\begin{aligned} \|u^{(n+1)}(t)\|_{B_{p,r}^s} &\leq \frac{\|u_0\|_{B_{p,r}^s}}{\sqrt[4]{1-4C\|u_0\|_{B_{p,r}^s}^2 t}} + \frac{C}{\sqrt[4]{1-4C\|u_0\|_{B_{p,r}^s}^2 t}} \\ &\quad \times \int_0^t \sqrt[4]{1-4C\|u_0\|_{B_{p,r}^s}^2 \tau} \frac{\|u_0\|_{B_{p,r}^s}^3}{(1-4C\|u_0\|_{B_{p,r}^s}^2 \tau)^{\frac{3}{2}}} d\tau \\ &\leq \frac{\|u_0\|_{B_{p,r}^s}}{\sqrt[4]{1-4C\|u_0\|_{B_{p,r}^s}^2 t}} \left(1 + C \int_0^t \frac{\|u_0\|_{B_{p,r}^s}^2}{(1-4C\|u_0\|_{B_{p,r}^s}^2 \tau)^{\frac{5}{4}}} d\tau \right), \end{aligned}$$

which implies

$$\begin{aligned} \|u^{(n+1)}(t)\|_{B_{p,r}^s} &\leq \frac{\|u_0\|_{B_{p,r}^s}}{\sqrt[4]{1-4C\|u_0\|_{B_{p,r}^s}^2 t}} \left(1 + \frac{1}{\sqrt[4]{1-4C\|u_0\|_{B_{p,r}^s}^2 t}} - 1 \right) \\ &= \frac{\|u_0\|_{B_{p,r}^s}}{\sqrt[4]{1-4C\|u_0\|_{B_{p,r}^s}^2 t}}. \end{aligned}$$

Hence, one can see that

$$\|u^{(n+1)}(t)\|_{B_{p,r}^s} \leq \frac{\|u_0\|_{B_{p,r}^s}}{(1-4C\|u_0\|_{B_{p,r}^s}^2 t)^{1/2}}.$$

Therefore, $(u^{(n)})_{n \in \mathbb{N}}$ is uniformly bounded in $\mathcal{C}([0, T]; B_{p,r}^s)$. Using the Moser-type estimates (see Lemma 2.4 (ii)), one finds that

$$\begin{aligned} \|[(u^{(n)})^2 - (u_x^{(n)})^2] \partial_x m^{(n+1)}\|_{B_{p,r}^{s-3}} &\leq C \|m^{(n+1)}\|_{B_{p,r}^{s-2}} (\|u^{(n)}\|_{B_{p,r}^s}^2 + \|u_x^{(n)}\|_{B_{p,r}^{s-1}}^2) \\ &\leq C \|u^{(n+1)}\|_{B_{p,r}^s} \|u^{(n)}\|_{B_{p,r}^s}^2, \end{aligned}$$

and

$$\|u_x^{(n)}(m^{(n)})^2\|_{B_{p,r}^{s-3}} \leq C \|m^{(n)}\|_{B_{p,r}^{s-2}}^2 \|u^{(n)}\|_{B_{p,r}^s} \leq C \|u^{(n)}\|_{B_{p,r}^s}^3.$$

Hence, using the equation (T_n) , we have

$$\partial_t u^{(n+1)} \in \mathcal{C}([0, T]; B_{p,r}^{s-1})$$

uniformly bounded, which yields that the sequence $(u^{(n)})_{n \in \mathbb{N}}$ is uniformly bounded in $E_{p,r}^s(T)$.

Next we are going to show that

$$(u^{(n)})_{n \in \mathbb{N}} \text{ is a Cauchy sequence in } \mathcal{C}([0, T]; B_{p,r}^{s-1}).$$

In fact, according to (3.3), we obtain that, for all $n, l \in \mathbb{N}$,

$$\begin{aligned} &\{\partial_t + [(u^{(n+l)})^2 - (u_x^{(n+l)})^2] \partial_x\} (m^{(n+l+1)} - m^{(n+1)}) \\ &= g(u^{(n+l)}, u^{(n)}, m^{(n+l)}, m^{(n)}, m^{(n+1)}), \end{aligned}$$

where

$$\begin{aligned} &g(u^{(n+l)}, u^{(n)}, m^{(n+l)}, m^{(n)}, m^{(n+1)}) \\ &= [(u^{(n)} - u^{(n+l)})(u^{(n)} + u^{(n+l)}) - (u_x^{(n)} - u_x^{(n+l)})(u_x^{(n)} + u_x^{(n+l)})] \partial_x m^{(n+1)} \\ &\quad - 2u_x^{(n+l)}(m^{(n+l)} - m^{(n)})(m^{(n)} + m^{(n+l)}) + 2(u_x^{(n)} - u_x^{(n+l)})(m^{(n)})^2. \end{aligned}$$

Applying Lemma 2.2 again, then for every $t \in [0, T]$, we obtain

$$\begin{aligned} & e^{-C \int_0^t \|[(u^{(n+l)})^2 - (u_x^{(n+l)})^2](\tau)\|_{B_{p,r}^{s-1}} d\tau} \|(m^{(n+l+1)} - m^{(n+1)})(t)\|_{B_{p,r}^{s-3}} \\ & \leq \|m_0^{(n+l+1)} - m_0^{(n+1)}\|_{B_{p,r}^{s-3}} + C \int_0^t e^{-C \int_0^\tau \|[(u^{(n+l)})^2 - (u_x^{(n+l)})^2](\tau')\|_{B_{p,r}^{s-1}} d\tau'} \\ & \quad \times \|g(u^{(n+l)}, u^{(n)}, m^{(n+l)}, m^{(n)}, m^{(n+1)})\|_{B_{p,r}^{s-3}} d\tau, \end{aligned}$$

which is equivalent to

$$\begin{aligned} & e^{-C \int_0^t \|[(u^{(n+l)})^2 - (u_x^{(n+l)})^2](\tau)\|_{B_{p,r}^{s-1}} d\tau} \|(u^{(n+l+1)} - u^{(n+1)})(t)\|_{B_{p,r}^{s-1}} \\ & \leq \|u_0^{(n+l+1)} - u_0^{(n+1)}\|_{B_{p,r}^{s-1}} + C \int_0^t e^{-C \int_0^\tau \|[(u^{(n+l)})^2 - (u_x^{(n+l)})^2](\tau')\|_{B_{p,r}^{s-1}} d\tau'} \\ & \quad \times \|g(u^{(n+l)}, u^{(n)}, m^{(n+l)}, m^{(n)}, m^{(n+1)})\|_{B_{p,r}^{s-3}} d\tau. \end{aligned}$$

In the case of $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$, one can deduce that

$$\begin{aligned} & \|[(u^{(n)} - u^{(n+l)})(u^{(n)} + u^{(n+l)}) - (u_x^{(n)} - u_x^{(n+l)})(u_x^{(n)} + u_x^{(n+l)})]\partial_x m^{(n+1)}\|_{B_{p,r}^{s-3}} \\ & \leq C \|m^{(n+1)}\|_{B_{p,r}^{s-2}} (\|u^{(n+l)} - u^{(n)}\|_{B_{p,r}^{s-1}} \|u^{(n+l)} + u^{(n)}\|_{B_{p,r}^{s-1}} \\ & \quad + \|u_x^{(n+l)} - u_x^{(n)}\|_{B_{p,r}^{s-2}} \|u_x^{(n+l)} - u_x^{(n)}\|_{B_{p,r}^{s-2}}) \\ & \leq C \|u^{(n+l)} - u^{(n)}\|_{B_{p,r}^{s-1}} (\|u^{(n)}\|_{B_{p,r}^s}^2 + \|u^{(n+1)}\|_{B_{p,r}^s}^2 + \|u^{(n+l)}\|_{B_{p,r}^s}^2), \\ & \|u_x^{(n+l)}(m^{(n+l)} - m^{(n)})(m^{(n)} + m^{(n+l)})\|_{B_{p,r}^{s-3}} \\ & \leq C \|u^{(n+l)}\|_{B_{p,r}^s} \|m^{(n+l)} - m^{(n)}\|_{B_{p,r}^{s-3}} \|m^{(n+l)} + m^{(n)}\|_{B_{p,r}^{s-2}} \\ & \leq C \|u^{(n+l)} - u^{(n)}\|_{B_{p,r}^{s-1}} (\|u^{(n)}\|_{B_{p,r}^s}^2 + \|u^{(n+1)}\|_{B_{p,r}^s}^2 + \|u^{(n+l)}\|_{B_{p,r}^s}^2), \end{aligned}$$

and

$$\begin{aligned} & \|(u_x^{(n)} - u_x^{(n+l)})(m^{(n)})^2\|_{B_{p,r}^{s-3}} \leq C \|u^{(n+l)} - u^{(n)}\|_{B_{p,r}^{s-2}} \|m^{(n)}\|_{B_{p,r}^{s-2}}^2 \\ & \leq C \|u^{(n+l)} - u^{(n)}\|_{B_{p,r}^{s-1}} \|u^{(n)}\|_{B_{p,r}^s}^2. \end{aligned}$$

From this, one finds that

$$\begin{aligned} & \|g(u^{(n+l)}, u^{(n)}, m^{(n+l)}, m^{(n)}, m^{(n+1)})\|_{B_{p,r}^{s-3}} \\ & \leq C \|u^{(n+l)} - u^{(n)}\|_{B_{p,r}^{s-1}} (\|u^{(n)}\|_{B_{p,r}^s}^2 + \|u^{(n+1)}\|_{B_{p,r}^s}^2 + \|u^{(n+l)}\|_{B_{p,r}^s}^2). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} & e^{-C \int_0^t \|[(u^{(n+l)})^2 - (u_x^{(n+l)})^2](\tau)\|_{B_{p,r}^{s-1}} d\tau} \|(u^{(n+l+1)} - u^{(n+1)})(t)\|_{B_{p,r}^{s-1}} \\ & \leq \|u_0^{(n+l+1)} - u_0^{(n+1)}\|_{B_{p,r}^{s-1}} + C \int_0^t e^{-C \int_0^\tau \|[(u^{(n+l)})^2 - (u_x^{(n+l)})^2](\tau')\|_{B_{p,r}^{s-1}} d\tau'} \\ & \quad \times \|(u^{(n+l)} - u^{(n)})(\tau)\|_{B_{p,r}^{s-1}} (\|u^{(n)}(\tau)\|_{B_{p,r}^s}^2 + \|u^{(n+1)}(\tau)\|_{B_{p,r}^s}^2 + \|u^{(n+l)}(\tau)\|_{B_{p,r}^s}^2) d\tau. \end{aligned}$$

Since $(u^{(n)})_{n \in \mathbb{N}}$ is uniformly bounded in $E_{p,r}^s(T)$ and

$$u_0^{(n+l+1)} - u_0^{(n+1)} = S_{n+l+1}u_0 - S_{n+1}u_0 = \sum_{q=n+1}^{n+l} \Delta_q u_0,$$

then there exists a constant C_T independent of n and l such that for all $t \in [0, T]$

$$\|(u^{(n+l+1)} - u^{(n+1)})(t)\|_{B_{p,r}^{s-1}} \leq C_T \left(2^{-n} + \int_0^t \|(u^{(n+l)} - u^{(n)})(\tau)\|_{B_{p,r}^{s-1}} d\tau \right).$$

Arguing by induction with respect to the index n , one can easily prove that

$$\|u^{(n+l+1)} - u^{(n+1)}\|_{L_T^\infty(B_{p,r}^{s-1})} \leq \frac{(TC_T)^{n+1}}{(n+1)!} \|u^{(l)}\|_{L_T^\infty(B_{p,r}^s)} + C_T \sum_{k=0}^n 2^{-(n-k)} \frac{(TC_T)^k}{k!}.$$

Similarly $\|u^{(l)}\|_{L_T^\infty(B_{p,r}^s)}$ can be bounded independently of l , we conclude that there exist some new constant C'_T independent of n and l such that

$$\|u^{(n+l+1)} - u^{(n+1)}\|_{L_T^\infty(B_{p,r}^{s-1})} \leq C'_T 2^{-n}.$$

Hence $(u^{(n)})_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0, T]; B_{p,r}^{s-1})$. \square

Proof of Theorem 3.1. Thanks to Lemma 3.1, we obtain that $(u^{(n)})_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0, T]; B_{p,r}^{s-1})$, so it converges to some function $u \in \mathcal{C}([0, T]; B_{p,r}^{s-1})$. We now have to check that u belongs to $E_{p,r}^s(T)$ and solves the Cauchy problem (1.1). Since $(u^{(n)})_{n \in \mathbb{N}}$ is uniformly bounded in $L^\infty([0, T]; B_{p,r}^s)$ according to Lemma 3.1, the Fatou property for the Besov spaces (Proposition 2.2 iv)) guarantees that u also belongs to $L^\infty([0, T]; B_{p,r}^s)$.

On the other hand, as $(u^{(n)})_{n \in \mathbb{N}}$ converges to u in $\mathcal{C}([0, T]; B_{p,r}^{s-1})$, an interpolation argument ensures that the convergence holds in $\mathcal{C}([0, T]; B_{p,r}^{s'})$, for any $s' < s$. It is then easy to pass to the limit in the equation (T_n) and to conclude that u is indeed a solution to the Cauchy problem (1.1). Thanks to the fact that u belongs to $L^\infty([0, T]; B_{p,r}^s)$, the right-hand side of the equation

$$\partial_t m + (u^2 - u_x^2) \partial_x m = -2u_x m^2$$

belongs to $L^\infty([0, T]; B_{p,r}^{s-2})$. In particular, for the case $r < \infty$, Lemma 2.3 enables us to conclude that $u \in \mathcal{C}([0, T]; B_{p,r}^{s'})$ for any $s' < s$. Finally, using the equation again, we see that $\partial_t u \in \mathcal{C}([0, T]; B_{p,r}^{s-1})$ if $r < \infty$, and in $L^\infty([0, T]; B_{p,r}^{s-1})$ otherwise. Moreover, a standard use of a sequence of viscosity approximate solutions $(u_\epsilon)_{\epsilon > 0}$ for the Cauchy problem (1.1) which converges uniformly in

$$\mathcal{C}([0, T]; B_{p,r}^s) \cap \mathcal{C}^1([0, T]; B_{p,r}^{s-1})$$

gives the continuity of the solution u in $E_{p,r}^s(T)$. \square

4. BLOW-UP SCENARIO AND GLOBAL CONSERVATIVE PROPERTY

In this section, attention is now turned to blow-up issue. We first present a blow-up scenario.

Theorem 4.1. *Let $u_0 \in H^s$, $s > 5/2$, and $u(t, x)$ be the solution of the Cauchy problem (1.1) with life-span T . Then T is finite if and only if*

$$\liminf_{t \uparrow T} \left[\inf_{x \in \mathbb{R}} (mu_x(t, x)) \right] = -\infty.$$

Proof. Since the existence time T is independent of the choice of s , in view of Remark 3.2, we only need to consider the case $s = 3$ by utilizing a simple density argument. Multiplying Eq.(1.1) by m and integrating over \mathbb{R} with respect to x yield

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} m^2 dx &= - \int_{\mathbb{R}} (u^2 - u_x^2) m m_x dx - 2 \int_{\mathbb{R}} u_x m^3 dx \\ &= \frac{1}{2} \int_{\mathbb{R}} (u^2 - u_x^2)_x m^2 dx - 2 \int_{\mathbb{R}} u_x m^3 dx \\ &= - \int_{\mathbb{R}} u_x m^3 dx. \end{aligned}$$

Differentiating the first equation with regard to x , one finds that

$$\begin{aligned} m_{xt} &= -2u_{xx}m^2 - 6u_xmm_x - (u^2 - u_x^2)m_{xx} \\ &= -2um^2 + 2m^3 - 6u_xmm_x - (u^2 - u_x^2)m_{xx}. \end{aligned}$$

Then multiplying the above equation by m_x and integrating over \mathbb{R} with respect to x , it leads to

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} m_x^2 dx &= - \int_{\mathbb{R}} (u^2 - u_x^2)m_x m_{xx} dx - 2 \int_{\mathbb{R}} um^2 m_x dx - 6 \int_{\mathbb{R}} u_x mm_x^2 dx + 2 \int_{\mathbb{R}} m^3 m_x dx \\ &= \frac{1}{2} \int_{\mathbb{R}} (u^2 - u_x^2)_x m_x^2 dx + \frac{2}{3} \int_{\mathbb{R}} u_x m^3 dx - 6 \int_{\mathbb{R}} u_x mm_x^2 dx \\ &= -5 \int_{\mathbb{R}} u_x mm_x^2 dx + \frac{2}{3} \int_{\mathbb{R}} u_x m^3 dx. \end{aligned}$$

Therefore,

$$\frac{d}{dt} \int_{\mathbb{R}} (m^2 + m_x^2) dx = -10 \int_{\mathbb{R}} u_x mm_x^2 dx - \frac{2}{3} \int_{\mathbb{R}} u_x m^3 dx.$$

If mu_x is bounded from below on $[0, T) \times \mathbb{R}$, i.e., there exists $N > 0$ such that $mu_x \geq -N$ on $[0, T) \times \mathbb{R}$, then it is thereby inferred from the above estimate that

$$\frac{d}{dt} \int_{\mathbb{R}} (m^2 + m_x^2) dx \leq 10N \int_{\mathbb{R}} (m^2 + m_x^2) dx.$$

Applying Gronwall inequality then yields for $t \in [0, T)$

$$\|m\|_{H^1}^2 \leq \int_{\mathbb{R}} (m^2 + m_x^2) dx \leq e^{10NT} \int_{\mathbb{R}} (m_0^2 + m_{0x}^2) dx = e^{10NT} \|m_0\|_{H^1}^2. \quad (4.1)$$

Differentiating the first equation with regard to x twice, one finds that

$$\begin{aligned} m_{xxt} &= -2u_xm^2 - 4umm_x + 6m^2m_x - 6u_xm_x^2 - 6u_xmm_{xx} - 6u_{xx}mm_x \\ &\quad - 2u_xmm_{xx} - (u^2 - u_x^2)m_{xxx} \\ &= -2u_xm^2 - 10umm_x + 12m^2m_x - 6u_xm_x^2 - 8u_xmm_{xx} - (u^2 - u_x^2)m_{xxx}. \end{aligned}$$

Then multiplying the above equation by m_{xx} and integrating over \mathbb{R} with respect to x , we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} m_{xx}^2 dx \\ &= - \int_{\mathbb{R}} (u^2 - u_x^2) m_{xxx} m_{xx} dx - 8 \int_{\mathbb{R}} u_x m m_{xx}^2 dx - 2 \int_{\mathbb{R}} u_x m^2 m_{xx} dx \\ & \quad + 12 \int_{\mathbb{R}} m^2 m_x m_{xx} dx - 10 \int_{\mathbb{R}} u m m_x m_{xx} dx - 6 \int_{\mathbb{R}} u_x m_{xx} m_x^2 dx. \end{aligned}$$

For the right hand side of the above equation, integrating by parts one finds that

$$\begin{aligned} - \int_{\mathbb{R}} (u^2 - u_x^2) m_{xxx} m_{xx} dx &= \frac{1}{2} \int_{\mathbb{R}} (u^2 - u_x^2)_x m_{xx}^2 dx = \int_{\mathbb{R}} u_x m m_{xx}^2 dx, \\ -2 \int_{\mathbb{R}} u_x m^2 m_{xx} dx &= 2 \int_{\mathbb{R}} m_x (u_{xx} m^2 + 2u_x m m_x) dx \\ &= 4 \int_{\mathbb{R}} u_x m m_x^2 dx + 2 \int_{\mathbb{R}} (u - m) m^2 m_x dx \\ &= 4 \int_{\mathbb{R}} u_x m m_x^2 dx - \frac{2}{3} \int_{\mathbb{R}} u_x m^3 dx, \end{aligned}$$

and

$$\begin{aligned} -6 \int_{\mathbb{R}} u_x m_{xx} m_x^2 dx &= 2 \int_{\mathbb{R}} u_{xx} m_x^3 dx = 2 \int_{\mathbb{R}} u m_x^3 dx - 2 \int_{\mathbb{R}} m m_x^3 dx \\ &= -2 \int_{\mathbb{R}} u_x m m_x^2 dx - 4 \int_{\mathbb{R}} u m m_x m_{xx} dx \\ & \quad + 2 \int_{\mathbb{R}} m^2 m_x m_{xx} dx, \end{aligned}$$

Where we have used

$$2 \int_{\mathbb{R}} u m_x^3 dx = -2 \int_{\mathbb{R}} m (u_x m_x^2 + 2u m_x m_{xx}) dx,$$

and

$$-2 \int_{\mathbb{R}} m m_x^3 dx = 2 \int_{\mathbb{R}} m (m_x^3 + 2m m_x m_{xx}) dx.$$

Therefore,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} m_{xx}^2 dx &= -7 \int_{\mathbb{R}} u_x m m_{xx}^2 dx + 2 \int_{\mathbb{R}} u_x m m_x^2 dx - \frac{2}{3} \int_{\mathbb{R}} u_x m^3 dx \\ & \quad + 14 \int_{\mathbb{R}} m^2 m_x m_{xx} dx - 14 \int_{\mathbb{R}} u m m_x m_{xx} dx. \end{aligned}$$

And so

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} (m^2 + m_x^2 + m_{xx}^2) dx \\ &= -14 \int_{\mathbb{R}} u_x m m_{xx}^2 dx - 6 \int_{\mathbb{R}} u_x m m_x^2 dx - 2 \int_{\mathbb{R}} u_x m^3 dx \\ & \quad + 28 \int_{\mathbb{R}} m^2 m_x m_{xx} dx - 28 \int_{\mathbb{R}} u m m_x m_{xx} dx. \end{aligned}$$

If mu_x is bounded from below on $[0, T) \times \mathbb{R}$, i.e., there exists $N > 0$ such that $mu_x \geq -N$ on $[0, T) \times \mathbb{R}$, then applying (4.1) we can deduce that

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}} (m^2 + m_x^2 + m_{xx}^2) dx \\
& \leq 14N \int_{\mathbb{R}} (m^2 + m_x^2 + m_{xx}^2) dx + 28(\|m\|_{L^\infty}^2 + \|um\|_{L^\infty}) \int_{\mathbb{R}} |m_x m_{xx}| dx \\
& \leq 14N \int_{\mathbb{R}} (m^2 + m_x^2 + m_{xx}^2) dx + 28\|m\|_{H^1}^2 \int_{\mathbb{R}} (m^2 + m_x^2 + m_{xx}^2) dx \\
& \leq 14(N + 2e^{10NT}\|m_0\|_{H^1}^2) \int_{\mathbb{R}} (m^2 + m_x^2 + m_{xx}^2) dx.
\end{aligned}$$

For any $t \in [0, T)$, using Gronwall inequality again it leads to

$$\begin{aligned}
\|u\|_{H^4}^2 & \leq \|m\|_{H^2}^2 = \int_{\mathbb{R}} (m^2 + m_x^2 + m_{xx}^2) dx \\
& \leq \exp(14T(N + 2e^{10NT}\|m_0\|_{H^1}^2)) \int_{\mathbb{R}} (m_0^2 + m_{0x}^2 + m_{0xx}^2) dx \\
& = \exp(14T(N + 2e^{10NT}\|m_0\|_{H^1}^2)) \|u_0\|_{H^4}^2.
\end{aligned}$$

The above inequality and Sobolev's embedding theorem ensure that the solution $u(t, x)$ does not blow up in finite time.

On the other hand, if

$$\liminf_{t \uparrow T} \left[\inf_{x \in \mathbb{R}} (mu_x(t, x)) \right] = -\infty,$$

by the existence Theorem 3.1 of the local strong solution and Sobolev's embedding theorem, we infer that the solution will blow-up in finite time. The proof of Theorem 4.1 is thus complete. \square

In order to demonstrate a conservative property, let us consider the trajectory equation

$$\begin{cases} \frac{dq}{dt} = (u^2 - u_x^2)(t, q(t, x)) \\ q(0, x) = x. \end{cases} \quad (4.2)$$

For all $t > 0$, a simple computation shows that

$$\begin{aligned}
& \frac{\partial}{\partial t} [m(t, q(t, x)) q_x(t, x)] \\
& = [m_t(t, q) + m_x(t, q) q_t] q_x + m q_{xt} \\
& = q_x [m_t(t, q) + (u^2 - u_x^2) m_x(t, q)] + 2u_x m^2 q_x \\
& = q_x [m_t + (u^2 - u_x^2) m_x + 2u_x m^2] = 0.
\end{aligned}$$

Therefore, $m(t, q(t, x)) q_x(t, x)$ is independent of the time variable t . That is

$$m(t, q(t, x)) q_x(t, x) = m(0, x) = u_0(x) - u_{0xx}(x).$$

5. ANALYTICITY OF SOLUTIONS

In this section, we shall establish the following analyticity result.

Theorem 5.1. *Let u_0 be a real analytic function on \mathbb{R} . There exist an $\varepsilon > 0$ and a unique solution u of the Cauchy problem (1.1) that is analytic on $(-\varepsilon, \varepsilon) \times \mathbb{R}$.*

Note that it is sufficient to verify the conditions 1) and 2) in the statement of the abstract Cauchy-Kowalevski Theorem 2.1 for both $F(U, V)$ and $G(U, V)$ in the system (2.3) since neither F nor G depends on t explicitly. Note that u_0 is analytic by the assumption of Theorem 5.1, we can deduce that both $|||u_0|||_s$ and $|||u'_0|||_{s'}$ are bounded. Without loss of generality, we assume that there exist constants $M_0, M_1 > 0$ such that $|||u_0|||_s \leq M_0$, $|||u'_0|||_{s'} \leq M_1$, and so $|||u'_0|||_{s'} \leq cM_0/(s-s')$. In order to prove 1), for $0 < s' < s < 1$, the estimates in Lemma 2.5 and 2.6 imply the following bounds

$$\begin{aligned}
|||F(U, V)|||_{s'} &\leq c^2 |||U + u_0|||_s^2 |||V + u'_0|||_{s'} + \frac{2c^2}{3} |||V + u'_0|||_{s'}^3 + \frac{2c^2}{3} |||U + u_0|||_s^3 \\
&\quad + c^2 |||U + u_0|||_s |||V + u'_0|||_{s'}^2 \\
&\leq c^2 [(R + M_0)^2 (R + \frac{cM_0}{s-s'}) + \frac{2}{3} (R + \frac{cM_0}{s-s'})^3 \\
&\quad + \frac{2}{3} (R + M_0)^3 + (R + M_0)(R + \frac{cM_0}{s-s'})^2] \\
&\leq \frac{2c^2}{3} (2R + M_0 + \frac{cM_0}{s-s'})^3, \\
|||G(U, V)|||_{s'} &\leq 2c^2 |||U + u_0|||_s |||V + u'_0|||_{s'}^2 + \frac{4c^2}{3} |||U + u_0|||_s^3 + \frac{c^2}{3} |||V + u'_0|||_{s'}^3 \\
&\quad + \frac{c^3}{s-s'} |||U + u_0|||_s^2 |||V + u'_0|||_{s'} + \frac{c^3}{s-s'} |||V + u'_0|||_{s'} |||V + u'_0|||_{s'}^2 \\
&\leq 2c^2 (R + M_0)(R + \frac{cM_0}{s-s'})^2 + \frac{4c^2}{3} (R + M_0)^3 + \frac{c^2}{3} (R + \frac{cM_0}{s-s'})^3 \\
&\quad + \frac{c^3}{s-s'} (R + M_0)^2 (R + M_1) + \frac{c^3}{s-s'} (R + M_1)(R + \frac{cM_0}{s-s'})^2,
\end{aligned}$$

hence condition 1) holds.

Note that to verify the second condition it suffices to estimate

$$|||F(U_1, V) - F(U_2, V)|||_{s'}, |||F(U, V_1) - F(U, V_2)|||_{s'},$$

and

$$|||G(U_1, V) - G(U_2, V)|||_{s'}, |||G(U, V_1) - G(U, V_2)|||_{s'}.$$

Since

$$|||F(U_1, V_1) - F(U_2, V_2)|||_{s'} \leq |||F(U_1, V_1) - F(U_1, V_2)|||_{s'} + |||F(U_1, V_2) - F(U_2, V_2)|||_{s'},$$

and

$$|||G(U_1, V_1) - G(U_2, V_2)|||_{s'} \leq |||G(U_1, V_1) - G(U_1, V_2)|||_{s'} + |||G(U_1, V_2) - G(U_2, V_2)|||_{s'}.$$

Using this together with Lemma 2.5 and 2.6, we get the following estimates

$$\begin{aligned}
|||F(U_1, V) - F(U_2, V)|||_{s'} &\leq |||(V + u'_0)(U_1 - U_2)(U_1 + U_2 + 2u_0)|||_{s'} + |||\partial_x(1 - \partial_x^2)^{-1} [(U_1 - U_2)(V + u'_0)^2]|||_{s'} \\
&\quad + \frac{2}{3} |||\partial_x(1 - \partial_x^2)^{-1} \{(U_1 - U_2)[(U_1 + u_0)^2 - (U_1 + u_0)(U_2 + u_0) + (U_2 + u_0)^2]\}| |||_{s'} \\
&\leq 2c^2 (|||V + u'_0|||_{s'}^2 + |||U_1 + u_0|||_s^2 + |||U_2 + u_0|||_s^2) |||U_1 - U_2|||_s \\
&\leq 2c^2 [2(R + M_0)^2 + (R + \frac{cM_0}{s-s'})^2] |||U_1 - U_2|||_s,
\end{aligned}$$

$$\begin{aligned}
& |||F(U, V_1) - F(U, V_2)|||_{s'} \\
& \leq |||(U + u_0)^2(V_1 - V_2)|||_{s'} + \frac{1}{3} |||(V_1 - V_2)[(V_1 + u'_0)^2 - (V_1 + u'_0)(V_2 + u'_0) + (V_2 + u'_0)^2]|||_{s'} \\
& \quad + |||\partial_x(1 - \partial_x^2)^{-1}[(V_1 - V_2)(U + u_0)(V_1 + V_2 + 2u'_0)]|||_{s'} \\
& \quad + \frac{1}{3} |||(1 - \partial_x^2)^{-1}\{(V_1 - V_2)[(V_1 + u'_0)^2 - (V_1 + u'_0)(V_2 + u'_0) + (V_2 + u'_0)^2]\}| |||_{s'} \\
& \leq 2c^2(|||V_1 + u'_0|||_{s'}^2 + |||V_2 + u'_0|||_{s'}^2 + |||U + u_0|||_s^2)|||V_1 - V_2|||_s \\
& \leq 2c^2[2(R + \frac{cM_0}{s - s'})^2 + (R + M_0)^2]|||V_1 - V_2|||_s,
\end{aligned}$$

$$\begin{aligned}
& |||G(U_1, V) - G(U_2, V)|||_{s'} \\
& \leq |||(V + u'_0)^2(U_1 - U_2)|||_{s'} + \frac{2}{3} |||(U_1 - U_2)[(U_1 + u_0)^2 - (U_1 + u_0)(U_2 + u_0) + (U_2 + u_0)^2]|||_{s'} \\
& \quad + |||(V + u'_0)(U_1 - U_2)(U_1 + U_2 + 2u_0)|||_{s'} + |||(1 - \partial_x^2)^{-1}[(V + u'_0)^2(U_1 - U_2)]|||_{s'} \\
& \quad + \frac{2}{3} |||(1 - \partial_x^2)^{-1}\{(U_1 - U_2)[(U_1 + u_0)^2 - (U_1 + u_0)(U_2 + u_0) + (U_2 + u_0)^2]\}| |||_{s'} \\
& \leq (3c^2 + \frac{c^3}{2(s - s')^2})(|||V + u'_0|||_{s'}^2 + |||V + u'_0|||_s^2 + |||U_1 + u_0|||_s^2 + |||U_2 + u_0|||_s^2)|||U_1 - U_2|||_s \\
& \leq (3c^2 + \frac{c^3}{2(s - s')^2})[2(R + M_0)^2 + (R + M_1)^2 + (R + \frac{cM_0}{s - s'})^2]|||U_1 - U_2|||_s,
\end{aligned}$$

and

$$\begin{aligned}
& |||G(U, V_1) - G(U, V_2)|||_{s'} \\
& \leq |||(V_1 - V_2)(U + u_0)(V_1 + V_2 + 2u'_0)|||_{s'} + |||(V_1 - V_2)(V_1 + u'_0)_x(V_1 + V_2 + 2u'_0)|||_{s'} \\
& \quad + |||(1 - \partial_x^2)^{-1}[(V_1 - V_2)(U + u_0)(V_1 + V_2 + 2u'_0)]|||_{s'} + |||(V_2 + u'_0)^2(V_1 - V_2)_x|||_{s'} \\
& \quad + \frac{1}{3} |||\partial_x(1 - \partial_x^2)^{-1}\{(V_1 - V_2)[(V_1 + u'_0)^2 - (V_1 + u'_0)(V_2 + u'_0) + (V_2 + u'_0)^2]\}| |||_{s'} \\
& \quad + |||(V_1 - V_2)_x(U + u_0)^2|||_{s'} \\
& \leq (4c^2 + \frac{c^3}{2(s - s')^2})(|||V_1 + u'_0|||_s^2 + |||V_1 + u'_0|||_{s'}^2 + |||V_2 + u'_0|||_{s'}^2 + |||U + u_0|||_s^2)|||V_1 - V_2|||_s \\
& \leq (4c^2 + \frac{c^3}{2(s - s')^2})[(R + M_0)^2 + (R + M_1)^2 + 2(R + \frac{cM_0}{s - s'})^2]|||V_1 - V_2|||_s.
\end{aligned}$$

This implies that the condition 2) also holds. Hence, the proof of Theorem 5.1 is complete.

6. NON-EXISTENCE OF SMOOTH TRAVELING WAVES

In this section, we prove that equation (1.1) does not have nontrivial smooth traveling waves. Assume that

$$u(t, x) = \phi(x - ct), \quad c \in \mathbb{R} \quad (6.1)$$

is a smooth traveling wave solution of (1.1). Then we have the following result.

Theorem 6.1. *There is no nontrivial smooth traveling wave solution $u(t, x) = \phi(x - ct)$, $c \in \mathbb{R}$ of (1.1) in $C([0, \infty); H^3(\mathbb{R})) \cap C^1([0, \infty); H^2(\mathbb{R}))$.*

Proof. We use a contradiction argument. Assume that $\phi \in H^3$ is a strong solution of (1.1). Then we have

$$c(\phi - \phi'')' = ((\phi^2 - \phi_x^2)(\phi - \phi''))' \text{ in } L^2(\mathbb{R}).$$

Since $\phi \in H^3(\mathbb{R}) \subset C_0^2(\mathbb{R})$, we find that

$$c(\phi - \phi'') = (\phi^2 - \phi_x^2)(\phi - \phi'') \text{ in } H^1(\mathbb{R}). \quad (6.2)$$

Note that $\phi \not\equiv 0$ and $\phi, \phi', \phi'' \rightarrow 0$ as $|x| \rightarrow \infty$, it implies that $\phi - \phi'' \neq 0$. Otherwise, $\phi = c_1 e^x + c_2 e^{-x}$, which gives $\phi \equiv 0$, $x \in \mathbb{R}$. It then follows from (6.2) that

$$\phi^2 - \phi'^2 = c. \quad (6.3)$$

Let $|x| \rightarrow \infty$. Then $\phi, \phi' \rightarrow 0$. It yields from (6.3) that $c = 0$. Hence we deduce from (6.3) that

$$\phi^2 - \phi'^2 = 0,$$

which implies that either $\phi = c_1 e^x$ or $\phi = c_2 e^{-x}$. This is a contradiction with $\phi \not\equiv 0$. This completes the proof of Theorem 6.1. \square

7. APPENDIX A. GEOMETRIC DESCRIPTIONS

It has been known for long time that integrable equations solved by the inverse scattering transform method have elegant geometric interpretations. Several different geometric frameworks have been utilized to provide geometric interpretations to integrable systems. For instance the celebrated CH equation was shown to describe the geodesic flow of the Riemannian metric on the diffeomorphism group of the circle [25] and pseudo-spherical surface [35]. It also arises from a non-stretching invariant planar curve motion in the centro-equiaffine geometry [6]. What is more, the mKdV equation, the sine-Gordon equation, the Schrödinger equation, the KdV equation and the Sawada-Kotera equation arise naturally from non-stretching invariant curve flows in Klein geometries (see [6, 7, 19, 20] and references therein).

In this appendix, we show that the equation (1.1) arises from non-stretching invariant curve flows respectively in two-dimensional Euclidean geometry and two-dimensional sphere, and it also describes a pseudo-spherical surface.

First, we study non-stretching invariant plane curve flows in the Euclidean geometry \mathbb{R}^2 , governed by

$$\frac{\partial \gamma}{\partial t} = f \mathbf{n} + g \mathbf{t}, \quad (\text{A.1})$$

where \mathbf{t} and \mathbf{n} are the Euclidean tangent and normal vectors, f and g are respectively the normal and tangent velocities depending on the curvature and its derivative with respect to the arc-length s of the curve. Let $ds = h dp$, where p is the free parameter independent of time t and h is the metric of the curve. A simple computation gives

$$h_t = (g_s - \kappa f)h.$$

Assume that the distance (along the curve) between any two points of the curve is invariant under the curve motion (A.1), that means $[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}] = 0$. Hence we deduce that

$$g_s - \kappa f = 0.$$

Let L be the parameter for a closed curve. A direct computation shows

$$\frac{\partial L}{\partial t} = \oint_{\gamma} (g_s - \kappa f) ds = - \oint_{\gamma} \kappa f ds.$$

Furthermore, assume that L is invariant under the curve flow (A.1). Then we require

$$\oint_{\gamma} \kappa f ds = 0.$$

By the curve flow (A.1), a straightforward computation leads to the evolution of the frame given by

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix}_t = \begin{pmatrix} 0 & f_s + \kappa g \\ -(f_s + \kappa g) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix}. \quad (\text{A.2})$$

Let θ be the angle between tangent vector of the curve and a fixed direction. Then $\mathbf{t} = (\cos \theta, \sin \theta)$, $\mathbf{n} = (-\sin \theta, \cos \theta)$. From (A.2), we get

$$\theta_t = f_s + \kappa g.$$

Hence the curvature $\kappa = \frac{d\theta}{ds}$ satisfies [19]

$$\kappa_t = (f_s + \kappa g)_s = \Omega f, \quad (\text{A.3})$$

where $\Omega = \partial_s^2 + \kappa^2 + \kappa_s \partial_s^{-1} \kappa$ is the recursion operator of the mKdV equation

$$\kappa_t = \kappa_{sss} + \frac{3}{2} \kappa^2 \kappa_s.$$

Set $f = -2v_s$, $\kappa = m \equiv v - v_{ss}$. Then $g = -(v^2 - v_s^2) + b$, b is a constant. Hence $v(t, s)$ satisfies the equation

$$m_t + [(v^2 - v_s^2)m]_s + (b + 2)v_{sss} - bv_s = 0. \quad (\text{A.4})$$

After the transformations $t \rightarrow t$, $s \rightarrow x = s + (b + 2)t$, then (A.4) becomes [34]

$$m_t + [(v^2 - v_x^2)m]_x + 2v_x = 0, \quad m = v - v_{xx}. \quad (\text{A.5})$$

Furthermore, by the scaling transformations $v \rightarrow \epsilon^{-1}u$, $t \rightarrow \epsilon^2\tau$, it then follows from (A.5) that

$$m_\tau + [(u^2 - u_x^2)m]_x + 2\epsilon^2 u_x = 0, \quad m = u - u_{xx}. \quad (\text{A.6})$$

Assume that u_x is uniformly bounded in \mathbb{R} and let $\epsilon \rightarrow 0$. Consequently, we arrive at (A.1).

Next, we consider the non-stretching curve flows on the two-dimensional sphere $S^3(R)$, governed by

$$\gamma_t = f\hat{n} + g\hat{t}, \quad (\text{A.6})$$

where \hat{n} and \hat{t} are respectively the normal and tangent vectors, f and g stand for the normal and tangent velocities, depending on the geodesic curvature ϕ of the curve and its derivative with respect to the arc-length s , they satisfy the Frenet equations [4]

$$\begin{pmatrix} \hat{r} \\ \hat{t} \\ \hat{n} \end{pmatrix}_s = \begin{pmatrix} 0 & \rho & 0 \\ -\rho & 0 & \kappa \\ 0 & -\kappa & 0 \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{t} \\ \hat{n} \end{pmatrix}, \quad (\text{A.7})$$

where $\rho = 1/R$ and $\hat{r} = \rho\gamma$ is the unit vector in the radial direction.

Since the Frenet frame is orthonormal, its time evolution is given by

$$\begin{pmatrix} \hat{r} \\ \hat{t} \\ \hat{n} \end{pmatrix}_t = \begin{pmatrix} 0 & \rho V & \rho U \\ -\rho V & 0 & A \\ -\rho U & -A & 0 \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{t} \\ \hat{n} \end{pmatrix}. \quad (\text{A.8})$$

Assume that the curve does not stretch during the curve motion, the arc-length does not depend on time. So s and t can serve as local coordinates on the sphere, and the commute relation

$$\left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right] = 0 \quad (\text{A.9})$$

holds. It follows from (A.7), (A.8) and (A.9) that

$$A = U_s + \phi V$$

and the curvature ϕ satisfies the equation

$$\phi_t = (U_s + \phi V)_s + \rho^2 U \quad (\text{A.10})$$

with the condition

$$V_s = \phi U. \quad (\text{A.11})$$

The substitution of (A.11) into (A.10) leads to the equation for the curvature

$$\phi_t = (\Omega + \rho^2)U, \quad (\text{A.12})$$

where Ω is the recursion operator of the mKdV equation. Set $\phi = m \equiv u - u_{ss}$, $U = -2u_s$, then $V = -(u^2 - u_s^2) + b$, and $u(t, s)$ satisfies the equation

$$m_t + [(u^2 - u_s^2)m]_s + (b + 2)u_{sss} - (b - 2\rho^2)u_s = 0. \quad (\text{A.13})$$

After the transformations $t \rightarrow t$, $s \rightarrow y = s + (b + 2)t$, this equation reduces to [34]

$$m_t + [(u^2 - u_y^2)m]_y + 2(1 + \rho^2)u_y = 0, \quad m = u - u_{yy}. \quad (\text{A.14})$$

Hence following the approximate argument again in (A.5) and (A.6), we obtain (A.1) in a different way.

Remark A.1. *It was shown by Reyes [35] that the CH and HS equations describe pseudo-spherical surfaces. Similarly, we can show that the equation (1.1) also describes pseudo-spherical surfaces, i.e., there exist one-forms*

$$\begin{aligned} \omega_1 &= \left[\sqrt{\frac{1-\lambda}{1+\lambda}} - \frac{1}{2}(1+\lambda)\sqrt{1-\lambda^2} + \left(\frac{\lambda}{1+\lambda} - \frac{1}{4}\lambda(1+\lambda) \right) m \right] dx \\ &\quad - \left[2\lambda^{-2} \sqrt{\frac{1-\lambda}{1+\lambda}} + \lambda^{-2}(1+\lambda)\sqrt{1-\lambda^2} + \left(\frac{2}{1+\lambda} + \frac{1+\lambda}{2\lambda} \right) (u_x + \lambda^{-1}u) \right. \\ &\quad \left. + \left(\frac{1}{4}(1+\lambda)\sqrt{1-\lambda^2} + \sqrt{\frac{1-\lambda}{1+\lambda}} + \left(\frac{1}{4}\lambda^{-2}(1+\lambda) + \frac{\lambda}{1+\lambda} \right) m \right) (u^2 - u_x^2) \right], \\ \omega_2 &= \lambda dx - [2\lambda^{-1} - 2\lambda^{-1}\sqrt{1-\lambda^2}u_x + \lambda(u^2 - u_x^2)]dt, \\ \omega_3 &= \left[-\sqrt{\frac{1-\lambda}{1+\lambda}} - \frac{1}{2}(1+\lambda)\sqrt{1-\lambda^2} - \left(\frac{\lambda}{1+\lambda} + \frac{1}{4}\lambda(1+\lambda) \right) m \right] dx \\ &\quad + \left[2\lambda^{-2} \sqrt{\frac{1-\lambda}{1+\lambda}} - \lambda^{-2}(1+\lambda)\sqrt{1-\lambda^2} + \left(\frac{2}{1+\lambda} - \frac{1+\lambda}{2\lambda} \right) (u_x + \lambda^{-1}u) \right. \\ &\quad \left. - \left(\frac{1}{4}(1+\lambda)\sqrt{1-\lambda^2} - \sqrt{\frac{1-\lambda}{1+\lambda}} + \left(\frac{1}{4}\lambda^{-2}(1+\lambda) - \frac{\lambda}{1+\lambda} \right) m \right) (u^2 - u_x^2) \right], \end{aligned}$$

which satisfy the structure equations for pseudo-spherical surface

$$d\omega_1 = \omega_3 \wedge \omega_2, \quad d\omega_2 = \omega_3 \wedge \omega_1, \quad d\omega_3 = \omega_1 \wedge \omega_2.$$

Based on the structure equations, using the equations for pseudo-potential, we are able to obtain two sets of conservation laws of equation (1.1).

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YING FU

DEPARTMENT OF MATHEMATICS, NORTHWEST UNIVERSITY, XI'AN, 710069, P. R. CHINA

E-mail address: fuying@nwu.edu.cn

GUILONG GUI

DEPARTMENT OF MATHEMATICS, JIANGSU UNIVERSITY, ZHENJIANG, JIANGSU, 212013, P. R. CHINA, AND THE INSTITUTE OF MATHEMATICAL SCIENCES, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, HONG KONG

E-mail address: glgui@amss.ac.cn

YUE LIU

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, ARLINGTON, TX 76019-0408

E-mail address: yliu@uta.edu

CHANGZHENG QU

DEPARTMENT OF MATHEMATICS, NORTHWEST UNIVERSITY, XI'AN, 710069, P. R. CHINA, AND SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455

E-mail address: czqu@nwu.edu.cn